Mathematical Modelling of Variable Viscosity Hydromagnetic Boundary Layer Flow with Thermal Radiation and Newtonian Heating

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ABSTRACT

Magnetohydrodynamic (MHD) boundary layer flow for a variable viscosity fluid subject to thermal radiation and Newtonian heating has numerous applications in industry and engineering such as designing of cooling systems in electronic devices, cooling of nuclear reactors, solar energy harvesting, thermal insulation, heat exchangers and in geothermal reservoirs. Heat transfer by thermal radiation is of significance to engineering processes that occur at high temperature and plays a major role in designing of equipment used in Nuclear reactors, gas turbines and equipment for propelling air crafts, missiles, satellites, and rockets. In this study, we use the fourth-order Runge-Kutta method and the shooting technique to find the numerical solution to the equations of fluid flow governing the boundary layer flow of a varying viscosity electrically conducting fluid that is subjected to a constant magnetic field in the presence of thermal radiation and Newtonian heating. The graphical results depicting the effects of various thermophysical parameters on the velocity and temperature profiles of the fluid are presented and then discussed quantitatively. From the study, we note that the velocity of the fluid increases with the increase in the values of the magnetic parameter and variable viscosity parameter. Furthermore, the temperature of the fluid increases with an increase in the values of the magnetic field parameter, Brinkmann number and local Biot number and decreases with the increase in thermal radiation parameter and variable viscosity parameter.

Key Words: Magnetohydrodynamic, Boundary Layer Flow, Thermal Radiation.
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Nomenclature

(u, v)	Velocity components	ρ	Density
(x, y)	Coordinates	ψ	Stream function
T	Temperature	σ	Electrical conductivity
T_∞	Free stream temperature	μ	Dynamic viscosity
B_0	Constant applied magnetic field	U_∞	Free stream velocity
β	Thermal expansion coefficient	ϕ	Nanoparticle concentration
Bi	Local Biot number	Pr	Prandtl number
Ra	Thermal radiation parameter	a	Variable viscosity parameter
Ha	Magnetic field intensity parameter	Br	Brinkmann number
θ	Dimensionless temperature	η	Transverse distance

1. INTRODUCTION

Magnetohydrodynamic (MHD) boundary layer flow and thermal radiation problems are applicable to numerous engineering and industrial applications such as design of cooling systems used in electronic devices, solar energy collection, design of equipment used in propulsion of air crafts, missiles, satellites, and hypersonic flights among many other applications. Numerous studies have

been undertaken on MHD boundary layer flows of fluid under various conditions. Study by Ishak [1] on MHD boundary layer flow over an exponentially stretching sheet subject to thermal radiation reported decrease in local heat transfer rate with increasing values of magnetic and radiation parameters. In their investigation Sajid & Hayat [2] observed that the Prandtl number and radiation parameter had opposite effect on the temperature of the fluid on an exponentially stretching sheet. Aliakbar *et al.* [3] studying effect of thermal radiation on MHD flow of Maxwellian fluid noted decrease in temperature of the fluid with increase in radiation parameter and Prandtl number. The temperature of the fluid was also observed to increase with increasing Eckert number. Siddheshwar & Mahabaleswar [4] in conducting their study, they examined the effect of radiation and heat source on MHD flow of viscoelastic fluid over a stretching sheet. Makinde & Sibanda [5] in their study, they considered MHD mixed-convective flow and heat transfer past a vertical plate in a porous medium with constant wall suction. Chamkha & Aly [6] focused their study on MHD free convection flow of a nanofluid past a vertical permeable plate in the presence of heat source or sink whereas Aziz & Khan [7] conducted examination into Natural convective boundary layer flow of nanofluid past a convectively heated vertical plate. Uddin *et al.* [8] performed their study on MHD free convective boundary layer flow of a nanofluid past a flat vertical plate subject to Newtonian heating boundary condition. The study by Samad & Mansur-Rahman [9] examined the interaction of thermal radiation and unsteady MHD flow past a vertical porous plate. The plate was immersed in a porous medium. Makinde & Sibanda [5] looked into MHD mixed convective flow and heat transfer past a vertical plate dipped in a porous medium with constant wall suction whereas Md. Anwar Hossain & Munir [10] gave analysis of a 2-D mixed convection flow of viscous incompressible temperature dependent viscous fluid past a vertical plate. The research conducted by Fang [11] provided analysis of how fluid property variation influences boundary layers of a stretching surface while Mahmoud [12] explained how varying viscosity affects hydromagnetic boundary layer flow a long a continuously moving vertical plate subject to radiation. The study by M. Anwar Hossain *et al.* [13] demonstrated how radiation affects free convection flow of a varying viscosity fluid on a porous vertical plate. Poornima & Reddy [14] using a non-linear stretching sheet, they demonstrated steady free convective boundary layer flow of a radiating nanofluid in the presence of transverse magnetic field. Investigations by Kandasamy *et al.* [15] presented effects of thermal stratification due to solar radiation, Brownian motion and thermophoresis on MHD boundary layer flow of nanofluid. Motsumi & Makinde [16] numerically investigated boundary layer flow of a nanofluid over a permeable moving flat plate in the presence of thermal radiation and viscous dissipation. The studies by [17-19] which were based on first law analysis and thermodynamic point of view examined MHD boundary layer flow with heat transfer. Narusawa [20] restricting the study on second law of thermodynamics presented findings on entropy generation minimization. Bejan [21,22] basing the study on second law analysis in heat transfer and thermal design problems, he introduced the concept of entropy generation minimization in the study. The study by Makinde & Aziz [18] considered plane Poiseuille flow with asymmetric convective heat transfer.

1.1 Magnetohydrodynamic

Magnetohydrodynamic (MHD) was first founded by Hannes Alfvén (1908-1995). MHD as a discipline deals with the dynamics of conductive fluids in magnetic fields. These conductive fluids comprise of Liquid metals (gallium, mercury, and molten iron), plasmas (such as the solar atmosphere) and strong electrolytes. In MHD; as the magnetic field and the conducting fluid comes into contact, the electric current of density **j** is induced into the conducting fluid resulting in the induced magnetic field. The total field **B** (induced plus imposed) interacts with induced current resulting in Lorentz force $F = j \times B$**.** The applications of MHD are broad and they include; MHD pump, MHD propulsion, metallurgy, MHD generators, and MHD flow meters. MHD micropumps are used as microsyringes for diabetics [23]. MHD pumps are also used in fusion research to create high impact velocities and in the cooling of nuclear reactors by pumping sodium coolant in the reactor core. The MHD propulsion serves as an alternative to the use of mechanical propellers in propelling marine vessels such as military submarines. MHD propulsion overcomes the problem of cavitation noise associated with the movement of propellers which is advantageous in the military where stealth is important [24, 25]. Metallurgical applications of MHD include Electromagnetic stirring of liquid metals during the formation of alloys, magnetic damping, levitation of liquid metals and electrolysis of aluminium oxide to aluminium. MHD generators can work under extremely high temperatures compared to the traditional generators, they have no movable parts reducing chances of mechanical failure and they work by converting thermal or kinetic energy directly into electrical energy. MHD flowmeters can be applied in determining the rate of blood flow through blood vessels. The first use of MHD blood flowmeters was by Kolin [26]. In surgery they are used in determining amount of blood flowing in vessel before, in the course of surgery and after the surgery [27, 28].

1.2 Boundary layer flow

Boundary layer in fluid flow was first introduced by Ludwig Prandtl in 1904. Boundary layer flow is of importance in determining friction drag of bodies moving in fluids; such as viscous drag on aerodynamics (airplanes, rockets, and projectiles such as missiles), hydrodynamics (ships, submarines, and torpedoes), automobiles (motor vehicles) and engineering structures such as buildings and bridges. A boundary layer in fluid flow can be described as a thin layer of viscous fluid just neighbouring the surface of the wall in which the fluid has a zero velocity at the wall/plate and a free stream velocity u_0 far away from the plate. The zero velocity at the walls is due to the wetting or sticking of the fluid on the surface of the wall as a result of adhesive forces between the wall and the fluid. This condition is known as the 'no-slip condition'. The fluid above the surface of the plate is in motion with shearing happening between its layers. The shear stress happening between the surface of the plate and the first

moving layer of the fluid adjacent to the plate is known as the wall shear stress (T_W). The boundary layer thickness (δ) is a function of the Reynolds number and refers to the distance from the solid wall to the height above the surface of the wall where the velocity of the fluid is 99% of the free stream velocity u_0. In the boundary layer flow, the hydrodynamic and thermal boundary layers are of great significance. In the hydrodynamic boundary layer, the fluid velocity is zero at the plate and its value increases to a free stream value u_0 far away from the plate. In the thermal boundary layer, the temperature of the fluid varies from the wall temperature T_0 to the free stream value T_∞ far away from the wall. The fluid particles in contact with the solid wall acquire a temperature equal to that of the wall. If the wall temperature is higher compared to the rest of the fluid, the fluid particles in contact with the wall exchanges heat with those in the neighbouring layers leading to the development of a thermal gradient in the fluid. The understanding of the hydrodynamic and thermal boundary layer is of significance in fluid mechanics since velocity is an important component in mass, momentum and energy equations while temperature gradient in the thermal boundary layer influences heat transfer in the fluid.
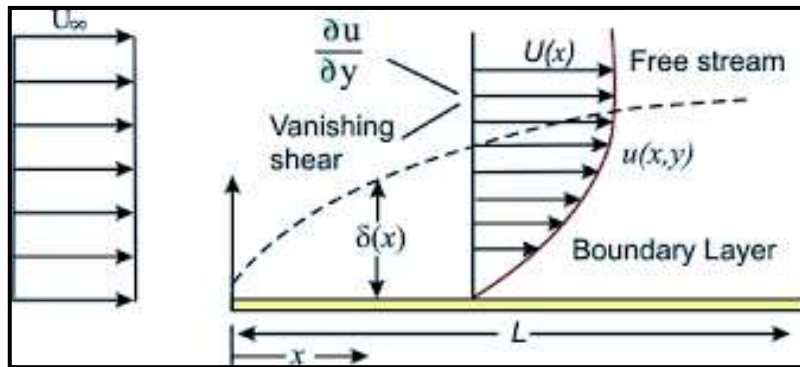

Figure 1: The boundary layer flow (Image Source: nptel.ac.in)

1.3 Thermal Radiation

Mechanisms by which heat energy gets transferred from one point to another are conduction, convection, and radiation. In thermal radiation heat is transferred in form of electromagnetic waves and it occurs at the speed of light ($c = 3.0 \times 10^8 ms^{-1}$). In thermal radiation heat energy can be propagated through space or vacuum (i.e. does not require material medium for transmission). It is the fastest means of heat transfer. The total radiant energy from a heated surface is arrived at by applying the Stefan-Boltzmann law. The Stefan-Boltzmann law states that; the rate of outward radiative energy per unit area emitted by an object of absolute temperature T is proportional to the fourth power of T. Mathematically Stefan-Boltzmann law is expressed as:

$$E = \epsilon \sigma^* T^4 \tag{1.1}$$

If the heated surface happens to be that of a blackbody (i.e. bodies considered to be perfect absorbers and perfect emitters), then $\epsilon = 1$ and equation (1.1) simplifies to;

$$E = \sigma^* T^4 \tag{1.2}$$

Equation (1.2) gives radiant energy per unit area of the blackbody. In equations (1.1) and (1.2) above, ϵ – refers to the emissivity of the surface, σ^* which is a constant (Stefan-Boltzmann constant) has a value of $\sigma^* = 5.670367 \times 10^{-8} Wm^{-2}K^{-4}$ and T stands for the absolute temperature of the surface-expressed in Kelvin (K). Heat transfer mechanism from the sun to the earth is by thermal radiation.
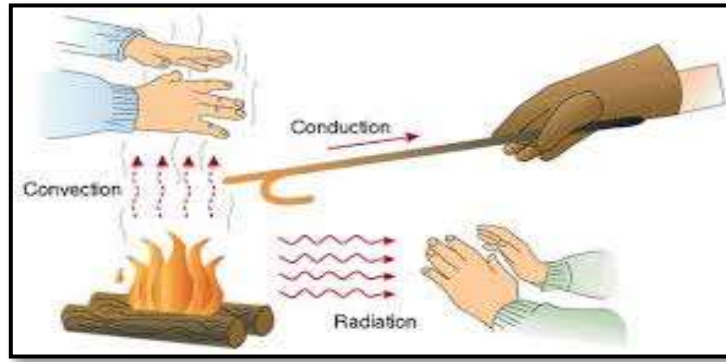
Figure 2: Modes of Heat Transfer (Image source: Green-Mechanic Website)

2. EQUATIONS GOVERNING FLUID DYNAMICS

2.1 The Continuity Equation

The continuity equation is obtained by considering conservation of mass for a system that states that mass can neither be created nor destroyed. To arrive at the continuity equation we apply the mass conservation principle on an infinitesimal volume of the fluid element within a moving fluid. The equation of continuity for the differential element in the Cartesian coordinate system is of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{Or} \quad \frac{\partial \rho}{\partial t} + \nabla . \left(\rho \vec{V}\right) = 0 \qquad (2.1)$$

Equation(2.1) is the Continuity equation for a compressible fluid in a rectangular Cartesian coordinate system. In equation (2.1) above ρ denotes density of the fluid, (u, v, w) refers to velocity component of the fluid in (x, y, z) directions respectively and $\nabla . \left(\vec{V}\right)$ refers to divergence of the velocity vector (i.e. the rate at which volume of a moving fluid element changes per unit volume.)

$$\text{We define} \quad \nabla \equiv \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \qquad (2.2)$$

If the flow of the fluid is steady (i.e. density does not vary with time), then $\frac{\partial \rho}{\partial t} = 0$ and equation (2.1) simplifies to:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{Or} \quad \nabla . \left(\rho \vec{V}\right) = 0 \qquad (2.3)$$

For incompressible flow (density is constant), the material derivative of density is zero (that is $\frac{\partial \rho}{\partial t} = 0$) and the continuity equation for incompressible flow becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{Or} \quad \nabla . \left(\vec{V}\right) = 0 \qquad (2.4)$$

2.2 Navier-Stokes (Momentum) Equation

Navier-Stokes Equation is obtained by considering Newton's second law of motion $\left(\vec{F} = m\vec{a} = m\frac{d\vec{u}}{dt}\right)$

Navier- Stokes equation for a flowing fluid is given by the equation

$$\frac{\partial V}{\partial t} + (V.\nabla)V = \frac{1}{\rho}[-\nabla P + \mu \nabla^2 V] + F \qquad (2.5)$$

Where F in the equation (2.5) above represents forces acting on flowing fluid.

If forces acting on flowing fluid are as a result of gravity, thermal expansion and the Lorentz force created by the magnetic field, then the Navier-Stokes equation (2.5) takes the form:

$$\frac{\partial V}{\partial t} + (V.\nabla)V = \frac{1}{\rho}[-\nabla P + \mu \nabla^2 V] + \rho \beta g \Delta T + \frac{1}{\rho}j \times B \qquad (2.6)$$

In equation(2.6) above; V denotes fluid's velocity, ρ represents the density of the fluid, P stands for pressure, μ denotes dynamic viscosity, g stands for gravitational force, β denotes thermal expansion coefficient, j represents electric current and B stands for

the magnetic field. If the fluid flow involves a nanofluid, then the quantities $\rho, \mu \ and \ \beta$ become $\rho_{nf}, \ \mu_{nf}$ and β_{nf} respectively which are defined as:

$$\rho_{nf=}(1 - \phi)\rho_f + \phi\rho_s \ ,$$

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \text{ , and} \qquad (2.7)$$

$$\beta_{nf} = (1 - \phi)\beta_f + \phi\beta_s$$

For a nanofluid equation (2.6)takes the form

$$\frac{\partial V}{\partial t} + (V.\nabla)V = \frac{1}{\rho_{nf}}\left[-\nabla P + \mu_{nf}\nabla^2 V\right] + (\rho\beta)_{nf}g\Delta T + \frac{1}{\rho_{nf}}j \times B \qquad (2.8)$$

2.3 Energy Equation.

The energy equation which is arrived at by applying the first law of thermodynamics (i.e. The amount of heat added to the system dQ is equal to change in internal energy dE plus the amount of energy lost due to work done on the system dW) i.e. $dQ = dE + dW$), takes the form:

$$\rho C_P \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q^* + \mu\Phi \qquad (2.9)$$

Where $\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2$

In the equation (2.9) above, (ρC_P) refers to heat capacitance of the fluid, (u, v, w) is the velocity component of the fluid in (x, y, z) directions respectively, T refers to local temperature of the fluid, k is the thermal conductivity of the fluid and q^* is the heat flux.

2.4 Magnetohydrodynamic (MHD) Flow

If the fluid flow happens to be in a magnetic field, then equations governing such a flow are momentum equation and Maxwell's equations of electromagnetism which are given as;

$$\nabla \times \boldsymbol{B} = \mu_0\boldsymbol{j} \qquad (2.10)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \qquad (2.11)$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \qquad (2.12)$$

Equation 2.10 is the Ampere's law; Equation 2.11 is the Faraday's law and equation 2.12 is the Ohm's law. In the above equations, μ_0 represents magnetic permeability,σ denotes electrical conductivity of the fluid, $\boldsymbol{j}$ represents electric current density, $\boldsymbol{E}$ stands for electric field and $\boldsymbol{B}$ is the magnetic field.

3. MATHEMATICAL FORMULATION

We consider a steady 2-D MHD boundary layer flow of a varying viscosity electrically conducting fluid with heat transfer over a horizontal plate placed in a stream of this fluid. The fluid is at the temperature T_∞ and is subjected to the magnetic field and thermal radiation. The lower surface of the plate is exposed to a heated fluid of temperature T_f that provides a heat transfer coefficient h_f. The fluid located on the upper side of the plate is subjected to Newtonian heating and a variation in the fluid property as a result of temperature is limited to viscosity. A constant magnetic field B_0 is imposed perpendicular to the flow. The induced magnetic field arising from flowing conductive fluid and the electric field due to the polarization of charges are considered negligible.
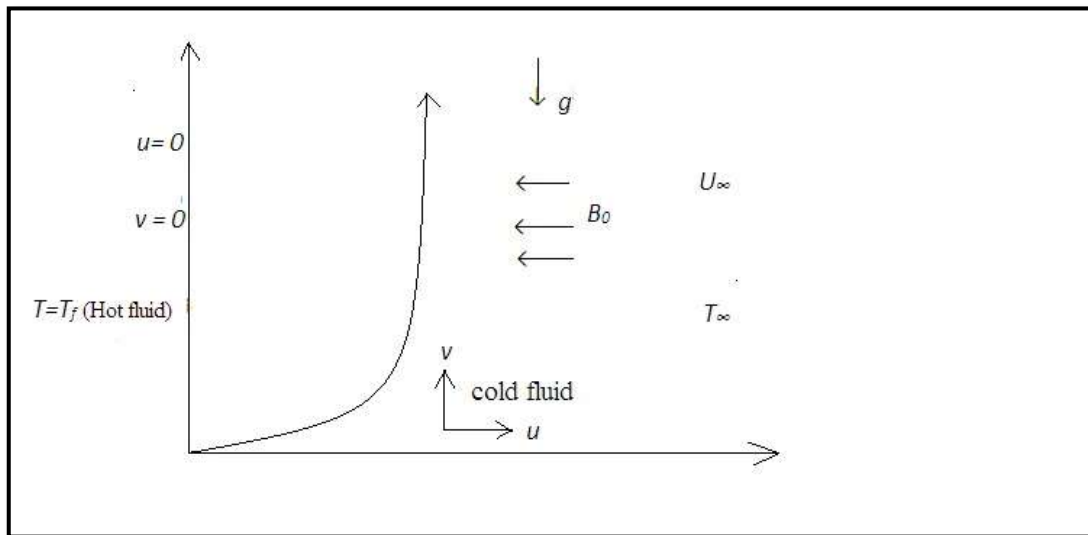
Figure 3: Flow configuration and coordinate system

The equations of fluid flow for the above flow configuration are;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + g\beta(T - T_\infty) - \frac{\sigma\beta_0^2}{\rho}(u - U_\infty) \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma\beta_0^2}{\rho c_p}(u - U_\infty)^2 \tag{3.3}$$

The governing boundary conditions are:

$$u(x,0) = 0, \quad v(x,0) = 0, \quad -k\frac{\partial T}{\partial y}(x,0) = h_f\left(T_f - T(x,0)\right)$$

$$u(x,\infty) = U_\infty, \quad T(x,\infty) = T_\infty \tag{3.4}$$

Where (u,v) denotes the fluid's velocity in (x,y) directions respectively, U_∞ refers to the free stream velocity of the fluid, c_p is the specific heat at constant pressure, T represents temperature, T_∞ is the free stream temperature of the fluid, ρ denotes density of the fluid, σ denotes fluid's electrical conductivity, k stands for fluid's thermal conductivity, g represents gravitational acceleration and μ stands for fluid's dynamical viscosity.

The dynamical viscosity μ given by equation (3.5) is an inverse linear function of temperature, [29].

$$\mu(T) = \frac{\mu_\infty}{1+\gamma(T-T_\infty)} \tag{3.5}$$

In equation (3.5) μ_∞ refers to the viscosity of the cold fluid and γ represents a constant. From [30] the radiative heat flux simplifies to equation (3.6) upon application of Rosseland approximation for radiation.

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \tag{3.6}$$

In equation (3.6), σ^* refers to the Stephan-Boltzmann constant and k^* refers to the mass absorption coefficient. Writing T^4 using truncated Taylor series about T_∞ to be a linear function of T by letting temperature difference to be sufficiently small within the flow:

$$T^4 \approx 4T_\infty^3 T - 3T_\infty^4 . \tag{3.7}$$

Introducing the following dimensionless quantities and the stream function Ψ

$$\eta = y\sqrt{\frac{U_\infty}{vx}}, \qquad \Psi = \sqrt{vxU_\infty}\; f(\eta), \quad v = \frac{\mu_\infty}{\rho}, \qquad \theta(\eta) = \frac{T-T_\infty}{T_f-T_\infty} \tag{3.8}$$

Thermal conductivity of the fluid is taken to be a linear function of temperature and is written as:

$$k(T) = k_\infty\left(1 + \gamma(T - T_\infty)\right) \tag{3.9}$$

The equation of continuity (3.1) is satisfied by the stream function by defining $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial x \partial y} = 0 \tag{3.10}$$

Equations (3.1) to (3.4) are solved together with their boundary conditions as follows:

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta} . \frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \tag{3.11a}$$

$$v = \frac{\mu_\infty}{\rho} \tag{3.11b}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} . \frac{\partial \eta}{\partial y} = \left(\frac{U_\infty^3}{vx}\right)^{1/2} f''(\eta) \tag{3.11c}$$

$$v \frac{\partial u}{\partial y} = \frac{\mu_\infty}{\rho} \left(\frac{U_\infty^3}{vx}\right)^{1/2} f''(\eta) \tag{3.11d}$$

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = \mu \left(\frac{U_\infty^3}{vx}\right)^{1/2} f'''(\eta) \tag{3.11e}$$

Substituting equations 3.11 (a-e) into equation (3.2) and introducing parameters Ha and Gr as defined then simplifying gives

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2}(1 + a\theta)f \frac{d^2 f}{d\eta^2} - \frac{a}{(1+a\theta)} \frac{d\theta}{d\eta} \frac{d^2 f}{d\eta^2} + Gr(1 + a\theta)\theta - Ha(1 + a\theta)\left(\frac{df}{d\eta} - 1\right) = 0 \tag{3.12}$$

For equation (3.3) we proceed as follows:

$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}$ Making T the subject gives

$$T = \theta(\eta)(T_f - T_\infty) + T_\infty \tag{3.13a}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta} . \frac{\partial \eta}{\partial x} = -\frac{1}{2}y \left(\frac{U_\infty}{vx^3}\right)^{1/2} \theta'(\eta)(T_f - T_\infty) \tag{3.13b}$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta} . \frac{\partial \eta}{\partial y} = \left(\frac{U_\infty}{vx}\right)^{1/2} \left(\theta'(\eta)(T_f - T_\infty)\right) \tag{3.13c}$$

$$u \frac{\partial T}{\partial x} = -\frac{1}{2}y \left(\frac{U_\infty^3}{vx^3}\right)^{1/2} f'(\eta)\theta'(\eta)(T_f - T_\infty) \tag{3.13d}$$

$$v \frac{\partial T}{\partial y} = \frac{\mu_\infty}{\rho} \left(\frac{U_\infty}{vx}\right)^{1/2} \theta'(\eta)(T_f - T_\infty) \tag{3.13e}$$

Substituting equations (3.13) (a-e) into (3.3) and introducing parameters Pr, Br and Ha and then simplifying equation (3.3) becomes:

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{2}Pr\beta f \frac{d\theta}{d\eta} + \frac{Br\beta}{(1+a\theta)}\left(\frac{d^2 f}{d\eta^2}\right)^2 + \beta BrHa\left(\frac{df}{d\eta} - 1\right)^2 = 0 \tag{3.14}$$

The boundary conditions (3.4) are transformed as follows:

$u = U_\infty f'(\eta)$ transforms $u(x,0) = 0$ to $\frac{df}{d\eta}(0) = 0$ and $u(x,\infty) = U_\infty$ is transformed to $\frac{df}{d\eta}(\infty) = 1$.

$$v = -\frac{\partial \Psi}{\partial x} = \frac{1}{4}x^{-1}U_\infty(f(\eta))^2 \text{ transforms } v(x,0) = 0 \text{ to } \quad f(0) = 0.$$

$$T = T_\infty + \theta(\eta)(T_f - T_\infty) \text{ transforms } T(x,\infty) = T_\infty \text{ to } \quad \theta(\infty) = 0 \text{ and}$$

$$-k\frac{\partial T}{\partial y}(x,0) = h_f(T_f - T(x,0)) \text{ is transformed to } \frac{d\theta}{d\eta}(0) = Bi[\theta(0) - 1]$$

Therefore for equations (3.12) and (3.14) the boundary conditions are:

$$f(0) = 0, \quad \frac{df}{d\eta}(0) = 0, \quad \frac{d\theta}{d\eta}(0) = Bi[\theta(0) - 1], \quad \frac{df}{d\eta}(\infty) = 1, \quad \theta(\infty) = 0 \tag{3.15}$$

For the absence of thermal radiation $\beta = 3Ra/(3Ra + 4)$ or $\beta = 1$. In equations (3.12) and (3.14) the prime symbol denotes derivative with respect to η and $Ha = \frac{\sigma B_0^2 x}{\rho U_\infty}$ refers to local magnetic field parameter,

$Bi = \frac{h_f}{k}\sqrt{\frac{vx}{U_\infty}}$ is the local convective heat exchange parameter, $Br = \frac{\mu_\infty U_\infty^2}{k(T_f - T_\infty)}$ denotes the Brinkmann number, $Pr = \frac{v}{\alpha}$ stands for the Prandtl number, $a = \gamma\left(T_f - T_\infty\right)$ refers to the viscosity variation parameter, and $Ra = \frac{kk^*}{4\sigma^* T_\infty^3}$ is the thermal radiation parameter.

3.1 Numerical Solution

Numerical solutions are obtained by solving equations (3.12) and (3.14) subject to the boundary conditions (3.15) using the fourth-order Runge-Kutta integration scheme together with a shooting technique. The computations are performed using a MAPLE computer programme that uses symbolic and computational computer language MAPLE. The method entails transforming equations (3.12) and (3.14) that are of third order in f and second-order in θ into a system of first-order differential equations. The system of first-order ordinary differential equations are obtained by letting

$$f_1 = f, \quad f_2 = f', \quad f_3 = f'', \; f_4 = \theta, \; f_5 = \theta' \qquad (3.16)$$

Where prime denotes derivative with respect to η.

The set of higher-order non-linear boundary value problem with their respective boundary conditions are reduced to first-order differential equations with appropriate initial conditions as shown below:

$$f_1' = f_2 \qquad (3.17\text{a})$$

$$f_2' = f_3 \qquad (3.17\text{b})$$

$$f_3' = -\frac{1}{2}(1 + af_4)f_1 f_3 + \frac{a}{(1+af_4)}f_5 f_3 - Gr(1 + af_4)f_4 + Ha(1 + af_4)(f_2 - 1) \quad (3.17\text{c})$$

$$f_4' = f_5 \qquad (3.17\text{d})$$

$$f_5' = -\frac{1}{2}Pr\beta f_1 f_5 - \frac{Br\beta}{(1+af_4)}f_3^2 - \beta BrHa(f_2 - 1)^2 \qquad (3.17\text{e})$$

Subject to the initial conditions

$$f_1(0) = 0, \quad f_2(0) = 0, \quad f_5(0) = B_i[f_4(0) - 1], \quad f_2(\infty) = 1, \; f_4(\infty) = 0 \quad (3.18)$$

4. RESULTS AND DISCUSSION

Numerical computations were performed for values of the physical parameters involved namely; Thermal radiation parameter (Ra), Prandtl number (Pr), Magnetic field intensity parameter (Ha), Variable viscosity parameter (a), local Biot number (Bi), and Brinkmann number (Br). For illustration of the results, numerical values were plotted in figure 4.1 to 4.7 and a detailed description of the effects of the above parameters on velocity and temperature profile was done.

4.1 Effect of Parameter Variation on Velocity Profiles

The graph in figure 4.1 depicts variation in velocity of fluid at different values of magnetic field parameter(Ha). From the graph the velocity of the fluid on the surface of plate is zero and it is attributed to the no-slip condition. Increasing values of (Ha) increases the velocity of the fluid to the free stream value U_∞ far away from the surface of the plate. Magnetic field imposed perpendicular to the flow generates Lorentz force in the fluid that opposes the motion of the fluid making the velocity to overshoot towards the surface of the plate. The graph in figure 4.2 gives velocity of the fluid at different values of variable viscosity parameter(a). Increasing values of (a) (i.e. as the viscosity of fluid decreases) leads to increase in fluid velocity as indicated by the figure. The decline in fluid's viscosity makes momentum boundary layer to decrease increasing the velocity gradient of the fluid.
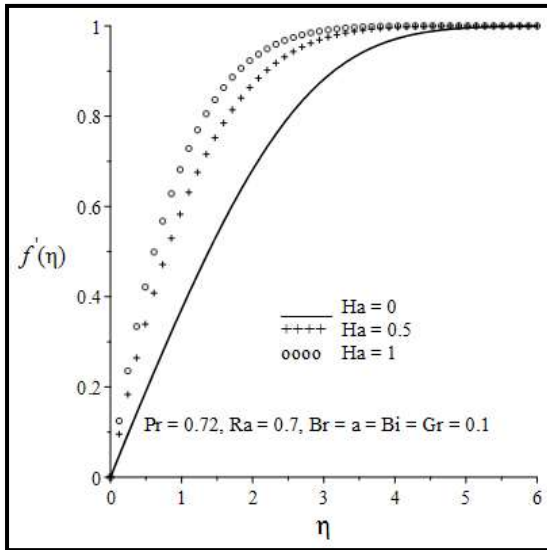
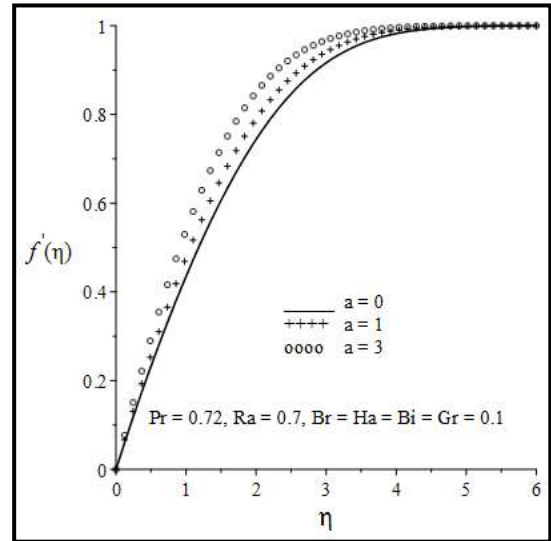**Figure 4.1: Velocity profile for varying *Ha***

Figure 4.2: Velocity profile for varying *a*

4.2 Effect of Parameter Variation on Temperature Profile

Graphs in figure 4.3-4.7 show variation of $\theta(\eta)$ for different values of Ha, Br, Bi, Ra and a. From the graphs the highest fluid temperature occurs on the surface of the plate. The free stream temperature decreases to zero value exponentially far away from the surface of the plate satisfying the given boundary conditions. In figure 4.3 as values of Ha increases, the fluid temperature also increases. Magnetic field imposed perpendicular to the flow produces Lorentz force that opposes the flow of the fluid increasing both temperature and thermal boundary layer of the fluid. This is attributed to the increased friction in the fluid. In figure 4.4 and 4.5 increasing values of Br and Bi results in increased fluid temperature in the thermal boundary layer due to generation of energy through viscous heating and Newtonian heating. In figures 4.6 and 4.7 there is a decrease in fluids temperature followed by a decrease in thermal boundary layer thickness as values of both Ra and a increases (i.e. as viscosity of the fluid decreases).
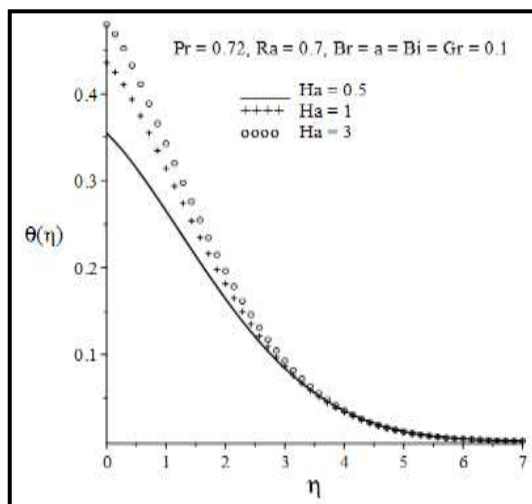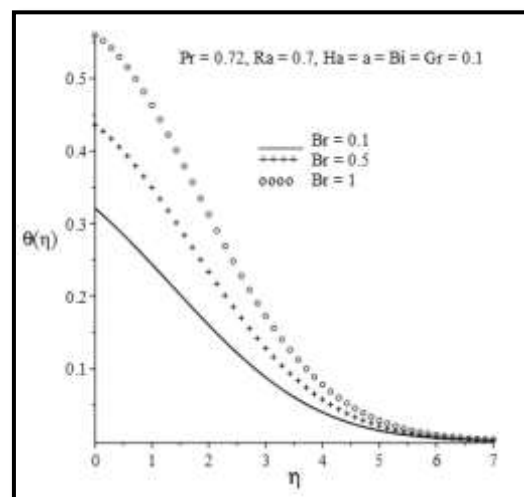

Figure 4.3: Temperature profile for varying *Ha*
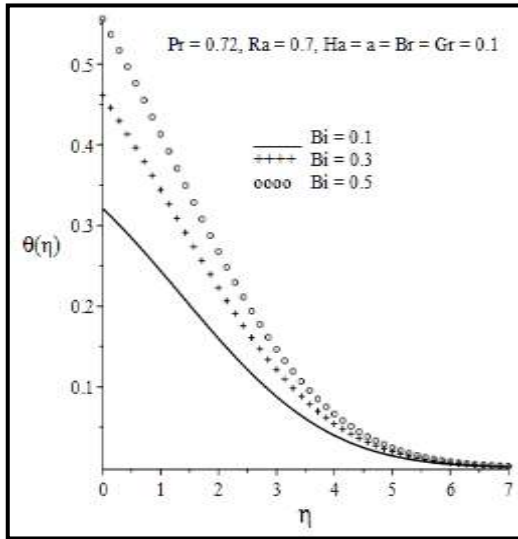
Figure 4.4: Temperature profile for varying *Br*

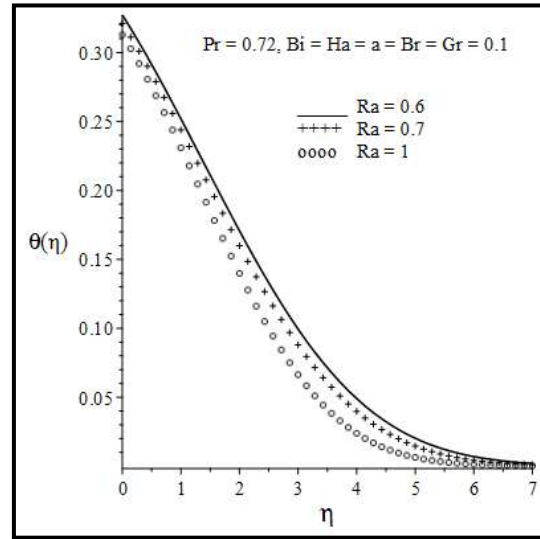**Figure 4.5: Temperature profile for varying *Bi***   **Figure 4.6: Temperature profile for varying *Ra***
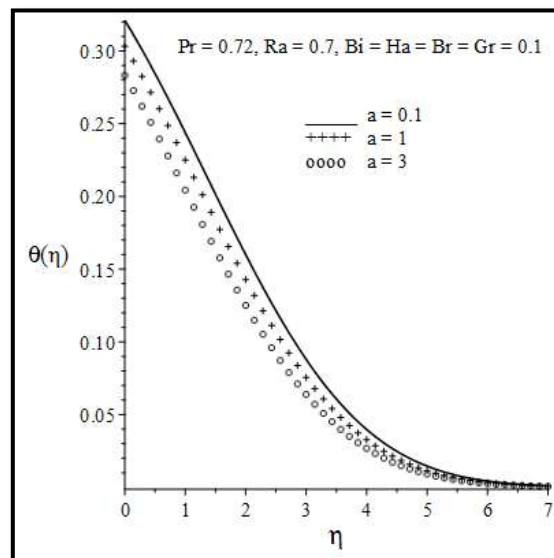

Figure 4.7: Temperature profile for varying *a*

5. CONCLUSIONS

In this study, we investigated steady 2-D MHD Boundary Layer Flow involving heat transfer over a horizontal plate in a stream of conducting fluid with varying viscosity at temperature T_∞ subject to thermal radiation and Newtonian heating. The model equations governing the flow were formulated and numerically solved using the shooting technique with a fourth-order Runge-Kutta integration scheme. The effects of thermal radiation parameter, magnetic field intensity parameter, variable viscosity parameter, local Biot number and Brinkman number on the velocity profile and temperature profile of the fluid were represented graphically. From the result we make the following conclusions:

i. Increase in values of magnetic parameter (Ha) and variable viscosity parameter (a) results in increased fluid velocity.
ii. The thickness of velocity boundary layer of the fluid decreases with Ha and a.
iii. The fluid's temperature increases with increase in values of Ha, Br, and Bi.
iv. Fluid's temperature decreases with increase in values of Ra and a.
v. Thermal boundary layer thickness increase with Ha, Br, Bi and decrease with increasing values of Ra and a.

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