MATCHING DOMINATION IN GRAPHS

P Bhaskarudu

Department of Mathematics

S. V. Arts College,

Tirupathi, Andhra Pradesh

India

ABSTRACT

A dominating set D is called a connected dominating set, if it induces a connected subgraph in G. Since a dominating set must contain atleast one vertex from every component of G, it follows that a connected dominating set for a graph G exists if and only if G is connected. The minimum of cardinalities of the connected dominating sets of G is called the connected domination number of G and is denoted by $\gamma_c(G)$*. We have defined new parameter called the matching dominating set and the matching domination number. We prove the following:*

- *If G is a graph without isolated vertices, then* $\gamma \leq \gamma_m$.
- *There exist bipartite graphs for which* $\gamma \neq \gamma_m$.
- *There exist graphs which are not bipartite for which* $\gamma = \gamma_m$
- $\gamma_m(K_p) = 2$
- $\gamma_m(K_{s,t}) = 2$
- *For any positive integer n,* $\gamma(K_{l,n}) = 1, \gamma_m(K_{l,n}) = 2$.
- *For any positive integer n,* $\gamma(W_n) = 1, \gamma_m(W_n) = 2$*, where* W_n *is a wheel on n vertices.*
- *For any positive integer n,* $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil, \gamma_m(P_n) = 2\left\lceil \frac{n}{4} \right\rceil$
- *For any positive integer n,* $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil, \gamma_m(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$.

KeyWords: *Dominating set, Domination number, Connected dominating set,Connected graph, Bipartite graph, Oddcycle.*

1. INTRODUCTION

Sampathkumar and Walikar[7] introduced the concept of connected domination in graphs. A dominating set D is called a connected dominating set, if it induces a connected subgraph in G.Since a dominating set must contain atleast one vertex from every component of G it follows that a connected dominating set for a graph G exists if and only if G is connected. The minimum of the cardinalities of the connected dominating sets of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. Sampathkumar and Walikar have obtained several interesting results on $\gamma_c(G)$. They have proved that

Theorem 1.1 $\gamma_c(K_p) = 1$

Theorem 1.2 $\gamma_c(K_p + G) = 1$, for any graph G.

Theorem 1.3 $\gamma_c(K_{m,n})$ = 1 when either m or n = 1;else 2 when m,n ≠ 1

Theorem 1.4 $\gamma_c(C_p)$ = p − 2

Theorem 1.5 For any tree T of order p, $\gamma_c(T)$ = p − e, where e is the number of pendent vertices i.e., vertices of degree 1 in T.

Theorem 1.6 $\gamma(G) \leq \gamma_c(G)$

Theorem 1.7 For any connected graph G(p,q) with maximum degree Δ, $\Delta^p \leq \gamma_c \leq 2q - p$.

Later, Hedetniemi and Laskar [10] improved and extended the results obtained by Sampathkumar and Walikar. Further,they have conjectured some relations concerning connected domination. Kulli and Sigarkanti [4] have studied the edge analogue of connected domination in graphs. They have obtained some exact bounds for the connected edge domination number. In all these results the inclcuded subgraph < S > constructed by the dominating set S of the graph plays a significant role. We have defined a new parameter called the matching dominating set and the matching domination number. These concepts of dominations are once again dependent on the induced subgraph induced by the dominating set.

2. MATCHING DOMINATION SET & MATCHING DOMINATION NUMBER

Definition 2.1 Let G be a simple graph with vertex set V and edge set E. A matching in G is a set M of edges of G such that every vertex of G is incident to atmost one edge in M. A matching M in G is called perfect matching if every vertex of G is incident to exactly one edge in M. If S ⊆ V we denote by G_s, the subgraph of G obtained by removing all the vertices in S and all of the edges to which vertices in S are incident. Denoting the number of components of G_s having an odd number of vertices by P(S), Tutte[8] has obtained the following remarkable result which gives a characterization for a graph to have a perfect matching. His result is, the graph G has a perfect matching if and only if P(S) ≤| S | ,for every subset S of V.

Definition 2.2 A graph H is called a subgraph of a graph G, if V (H) ⊆ V (G) and E(H) ⊆ E(G) For any set S of vertices of G, the induced subgraph < S > is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in < S > if and only if they are adjacent in G. The concept of domination in graphs was first introduced by Ore[10]. During the last three decades, the tremendous growth of research in this area has developed it, as a major area of research activity in graph theory.

The Dominating set is defined as follows

Definition 2.3 A set S ⊆ V is said to be a dominating set in a graph G if every vertex in V/S is adjacent to some vertex in S and the domination number 'γ' of G is defined to be the minimum cardinality of all dominating sets in G. We have introduced a new parameter called the matching domination set of a graph.

It is defined as follows:

Definition 2.4 A dominating set of a graph G is said to be matching dominating set if the induced subgraph < D > admits a perfect matching. The cardinality of the smallest matching dominating set is called matching domination number and is denoted by γ.

Illustration

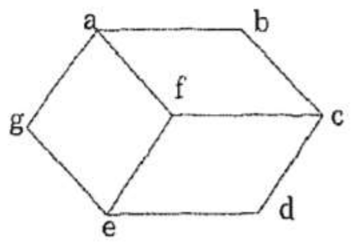

Figure 2.1

In this graph {a, b, c, f, e, g } is a matching domination set, since this is a dominating set and the induced subgraph {a, b, c, e, f, g} has perfect matching formed by the edges af, bc, eg, {a,b,e,f } is also matching dominating set. Similarly {a,b,c,g} is a matching dominating set where the induced subgraph of this set admits a perfect matching given by the edges be,ag. However there are no matching dominating sets of lower cardinality and it follows that the matching domination number of the graph in figure 2.1 is 4. Thus a graph can have many matching dominating sets of minimal cardinality. We make the following observations as an immediate consequence.

(a) Not all dominating sets are matching domination sets. For example in figure 2.1, {a,c,e} is a dominating set but it is not a matching dominating set.

(b) The cardinality of matching dominating set is always even.By definition 2.4, the matching dominating set D of a graph requires the admission of a perfect matching by the induced subgraph $< D >$. Thus it is necessary that D has even number of vertices for admitting a perfect matching.

(c) Not all dominating sets with even number of vertices are matching dominating sets. For example in figure 2.1,{b,d,g,f } is a dominating set containing even number of vertices, but induced subgraph formed by these four vertices does not have a perfect matching.

The following result is an immediate consequence of the definition 2.1.

(d) The necessary condition for a graph G to have matching dominating set is that G is a graph without isolated vertices. The matching domination number of the graph G (figure 2.1) is 4, whereas the domination number is 2 ; {a,d} being a minimal dominating set. If G is a graph with isolated vertices then any dominating set should include these isolated vertices and consequently the induced subgraph of this set containing isolated vertices will not admit a perfect matching.

In general, since every matching dominating set is a dominating set with an even number of vertices, we have the following result.

Theorem 2.5

If G is a graph without isolated vertices, then $\gamma \leq \gamma_m$.

Proof:

It follows from the definition 2.4 of matching dominating set and that the matching domination set should contain an even number of vertices, so as to allow a perfect matching in the induced subgraph, besides being a dominating set. Thus it follows that $\gamma \leq \gamma_m$. However it is interesting to note that there are some graphs for which $\gamma = \gamma_m$. We call a cycle on four vertices where at each vertex of the cycle has some pendent; vertices, a four cycle pendent. Consider the union of these types of graphs at nodes of the 4- cycles, we call it as union of 4-cycle-pendents. It is interesting to see that 4 cycle pendents have the property$\gamma = \gamma_m$.

In this context, we have the following result.

We prove the result by giving example of graphs (figure 2.2 & figure 2.3) with the required conditions.

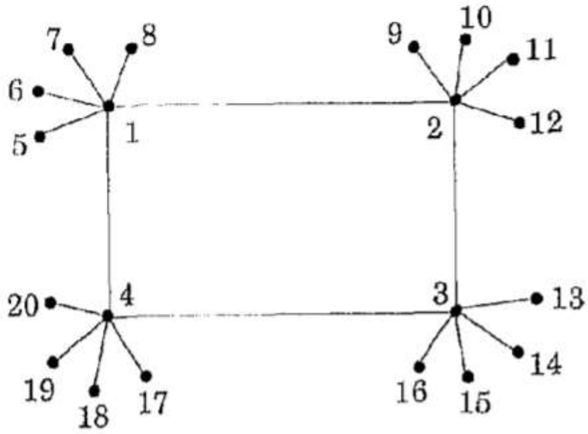

Figure 2.2 : 4-cyclic-pendent

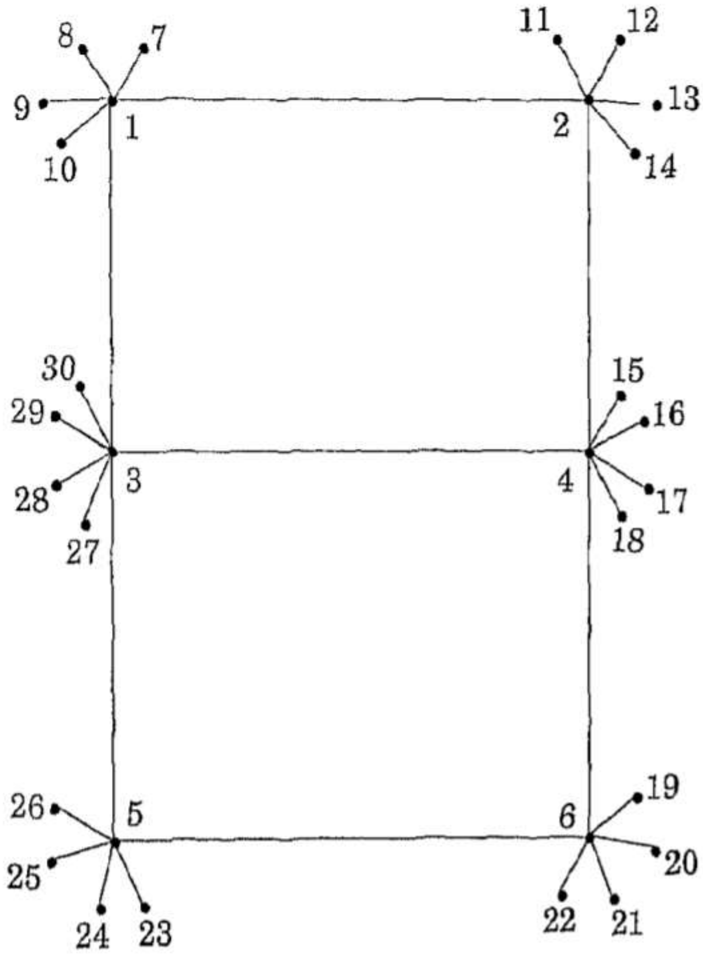

Figure 2.3 : Union of 4-cyclic-pendents

Theorem 2.6

(i) There exist bipartite graphs for which $\gamma \neq \gamma_m$.

(ii) There exist graphs which are not bipartite for which $\gamma = \gamma_m$.

PROOF:

(i) There are bipartite graphs for which $\gamma \neq \gamma_m$ Consider the following graph which is a path on 5 vertices. The graph(in figure 2.4) is a bipartite graph whose domination number is 2 with minimal dominating set u_2, u_4 and the matching domination number is 4. The minimal matching dominating set is u_1, u_2; u_4, u_5.

Thus γ.

(ii) Graphs which are not bipartite for $\gamma = \gamma_m$. Consider the graph(figure 2.5).

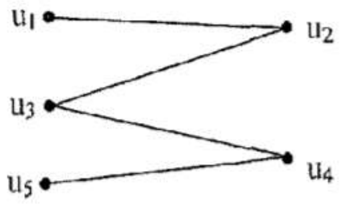

Figure2.4

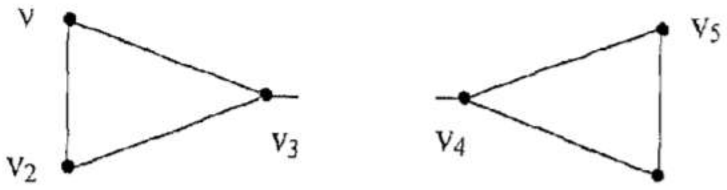

Figure2. 5

This is not a bipartite graph(figure 2.5) as there are odd cycles K_3. The domination number is 2 and minimum dominating set is $\{v_3, v_4\}$. The matching domination number is 2 and minimal matching domination set is $\{v_3, v_4\}$.

Theorem 2.7:

$\gamma_m(K_p) = 2$

Proof:

We know that for any +ve integer p, $\gamma_c(K_p) = 1$, as every vertex in K_p is adjacent to every other vertex in K_p.Further $\gamma_m(K_p) = 2$,as any pair of vertices will constitute a matching dominating set.

Theorem 2.8

$\gamma_m(K_{s,t}) = 2$

Proof:

$K_{s,t}$ is a complete bipartite graph with bipartition (X,Y). Any pair of vertices (u,v), u∈X, v ∈Y will constitute a dominating set as well as matching dominating set. Thus $\gamma(K_{s,t}) = \gamma_m(K_{s,t}) = 2$

Theorem 2.9

For any positive integer n, $\gamma(K_{1,n}) = 1, \gamma_m(K_{1,n}) = 2$.

Proof:

If $K_{l,n}$ is a bipartite graph with bipartition (X, Y). u∈X will be the only vertex in X and is adjacent to every vertex in Y. Hence {u} is minimal dominating set. $\gamma(K_{l,n}) = 1$ For any v∈Y, {u,v} will form a minimal matching dominating set. Therefore, $\gamma_m(K_{l,n}) = 2$.

Theorem 2.10

For any positive integer n,$\gamma(W_n) = 1, \gamma_m(W_n) = 2$, where W_n is a wheel on n vertices.

Proof:

{1} is minimal dominating set and {l,v},v∈{2, 3, 4, 5, 6} will constitute a minimal matching dominating set.

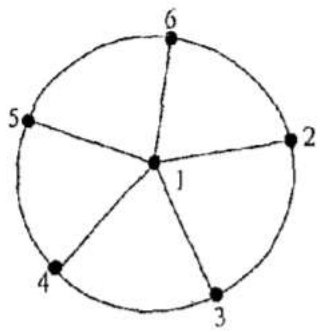

Fig .6

Theorem 2.11

For any positive integer n, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil, \gamma_m(P_n) = 2\left\lceil \frac{n}{4} \right\rceil$ where $\lceil X \rceil$ denotes least positive integer ≥ x.

Proof:

Let P_n, be a path on n vertices. We can label vertices of P, as {1, 2, 3, ..., n}. Divide them into subpaths of length not exceeding 4. We have the subpaths {1, 2, 3,4}, {5, 6, 7,8}, {9,10, 11, 12}, ... (...... n .... }. We will have four cases.

Case (i)

When n = 4t, then {2, 3, 6, 7, 10, 11, ... n-2, n-1} is minimal matching dominating set and {2, 3}, {6, 7}, {10, 11}, ... {n-2, n-1} forms a perfect matching of the induced graph of the minimal dominating set.

Case(ii)

When n= 4t+1, then {2, 3, 6, 7, ..., n-3, n-2, n-1, n } is minimal matching dominating set and {2, 3}, {6, 7}. .... {n- 3, n-2}, {n-1, n} forms a perfect matching of the induced graph of minimal dominating set.

Case(iii)

When n = 4t + 2, then {2, 3, 6 7, ... n-4, n-3, n-1, n} is minimal matching dominating set and {2, 3}, {6, 7}, ... {n- 4,n-3}, {n-1, n} forms a perfect matching of the induced graph of the matching dominating set.

Case(iv)

When n = 4t + 3 then {2, 3, 6, 7, 10, 11, ... n-5, n-4, n-1, n} is a minimal matching dominating set and {2, 3}, {6, 7}, ... {n-5, n-4}, {n-1, n} forms a perfect matching of the induced graph of the matching dominating set. The cardinality in each case is 2d n 4 e.

The following result follows similarly,

Theorem 2.12

For any positive integer n,$\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$,$\gamma_m(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$

3. CONCLUSION

The study of the product graphs,the matching domination of product graphs has been providing us sufficient stimulation for obtaining some in-depth knowledge of the various properties of the graphs. It is hoped that encouragement provided by this study of these product graphs will be a good straight point for further research.

4. REFERENCE

[1]Bondy and Murthy, “Graph theory with applications”,*Macmillan*(1976).

[2] Harary.F, “Graph Theory”, *Addison-Wesley*, 1969. .

[3] Kulli. V.R. and Sigarkanti, S.C., “Connected edge domination number”, *Pre-print*.

[4] Kulli,V.R., and Janakiram, B., “The maximal domination number of a graph” ,*Graph theory Notes of Newyork* XXXIII,11-18(1979),607613.

[5] Laskar. R.C. and Walikar. H.B., “On domination related concepts in Graph Theory” in *: Lecture notes in Match.,* 885 981, 308-320.

[6] Lewin.K., “Field theory in Social Science” *New York, Harper*, 1951.

[7] Sampathkumar. E. and Walikar. H.B., “The connected domination number of a graph,” *J. Math. Phy. Sci.*, 13 (1979),607-613.

[8] Tutte, W.T., “The factorization of linear graphs*,” London Math.* Soc., 22 (1947), 107-111.

[9] Ore. O., “Theory of Graphs,”*American Math. Soc. Colloq. Publ., 38, Providence, RI,* (1962).

[10] Hedetniemi. S.T.,and Laskar. R., “Connected domination in graphs, in Graph Theory andCombinatorics” *(Ed. Bela Bollobas), Academic Press, London,* 1984.