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Coefficient Estimate for a Subclass of Analytic Functions Defined by a Generalized Differential Operator

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ABSTRACT

The purpose of this paper is to study the coefficient estimates of the class of functions in $S^{n,t}_{\mu,\beta}(q)$consisting of starlike functions. The sharp upper bounds for the initial coefficients and the Fekete-Szego functional of the functions in the class were established using the Opoola Differential Operator.

Key Words: Analytic Function, Coefficient Estimates, Fekete-Szego Functional, Hankel Determinant, Opoola Differential Operator, Univalent Function.

1. INTRODUCTION

Let A denote the class of function $f(z)$ analytic in the open unit disk $U = \{z \in C: |z| < 1\}$ and let $S \in A$ denote the class of analytic functions $f(z)$ in U which are univalent in U and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (1.1)$$

Let $S^{n,t}_{\mu,\beta}(q)$denote the class of analytic functions $f(z)$ in U which are normalized by $f(0) = 0$, $f'(0) = 1$. Given $f(z)$ and $g(z)$ to be analytic functions, then $f(z)$ is subordinate to $g(z)$, if and only if there exist a function $w(z)$ analytic in U such that $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. Therefore,

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$$

And,

$$f(|z| < 1)(|z| < 1)$$

1.1 Opoola differential Operator

Let A denote the class of functions $f(z)$ analytic and univalent in the Unit disk $U = \{z \in C: |z| < 1\}$, and have the form (1.1). The Opoola Differential Operator is defined as $D^n(\mu,\beta,t)f(z): A \to A$ with

$$D^0(\mu,\beta,t)f(z) = f(z)$$

$$D^1(\mu,\beta,t)f(z) = tzf^I(z) - z(\beta-\mu)t + (1 + (\beta - \mu - 1)t)f(z)$$

$$D^n(\mu,\beta,t)f(z) = D[D_t^{n-1}f(z)]$$

For the function in the form of (1.1)

$$D^n_{\mu,\beta,t}f(z) = z + \sum_{k=2}^{\infty} 1 + (k - \beta - \mu - 1)^n a^k z_k \qquad (1.2)$$

Definition 1.1. A function $f(z)$ is said to be subordinate to $g(z)$ where $f(z)$ and $g(z)$ are both analytic in the unit disk U and denoted as:

$$f(z) \prec g(z), \quad z \in U.$$

Such that

$$f(z) = g\big(\omega(z)\big), \qquad z \in U.$$

If the function $g(z)$ is univalent in U, then $f(z)$ is said to be subordinate to $g(z)$ if

$$f(0) = g(0), \text{ and } f(U) \subset g(U)$$

If there exist a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0, |\omega(z)| < 1$

Let us denote the function $\omega(z)$ by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k \qquad (1.3)$$

A function $f(z) \in A$ is said to be in class $S^{n,t}_{\mu,\beta}(q)$ if and only if

$$R_e \frac{D^{n+1}(\mu,\beta,t)f(z)}{D^n(\mu,\beta,t)f(z)} > 0, \qquad Z \in U$$

Let P be the class of functions $p(z)$ of the form

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

Which are analytic in the open disk $U = z \in C\colon |z| < 1$ and satisfying the condition

$$R_e\big(P(z)\big) > 0 \quad p \in U.$$

The class $S^{n,t}_{\mu,\beta}(q)$ generalizes the class $S^*_n(q)$ studied by Raina and Sokol. It also generalizes the work or Bello and Opoola (2017).

2. LITERATURE REVIIEW

Let $S^{n,t}_{\mu,\beta}(q)$ be the class of function analytic in the unit disk U and normalized by $f(0) = f^I(0) - 1 = 0$ and satisfying the condition.

$$\frac{D^{n+1}(\mu,\beta,t)f(z)}{D^n(\mu,\beta,t)f(z)} \prec \sqrt{1 + (z)^2} + (z) = q(z) \qquad z \in U \qquad (1.4)$$

Where the branch of square root is chosen to be $q(\omega(0)) = 1$

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It may be noted that the set $q(U)$ lies in the right half plane and it is not a starlike domain with respect to the origin

Raina and Sokol (2000) studied the class $S^*(q)$ and obtained the following result:

(I)Theorem :

If $f(z) = z + \sum_{k=2}^{\infty} a^k z_k$ then,

$$|a_2| \leq 1, \qquad |a_3| \leq \frac{3}{4}| \qquad |a_4| \leq \frac{1}{2}$$

(II)Theorem :If $f(z) = z + \sum_{k=2}^{\infty} a^k z_k$ then,

$$|a_3 - \lambda a_2^2| \leq \max\{a_3 - \lambda a_2^2\}$$

When $\lambda \in C$.

Fekete-Szego problem is when the upper estimate is obtained

Bello and Opoola (2017) also obtained the Second Hankel Determinant for the class $S^*(q)$.

(III)Theorem:

If $f(z) = z + \sum_{k=2}^{\infty} a^k z_k$ then,

$$|a_2 a_4 - a_3^2| \leq$$

In recent years the Upper estimates of determinant $H_{q(n)}$ has been given attention to, the Fekete Szego estimate $H_{2(1)} = |a_3 - a_2^2|$ and $H_{2(2)} = |a_2 a_4 - a_3^2|$ have been greatly studied.

3. OBJECTIVES:

In this work the coefficient bounds, fekete-Szego estimates and Second Hankel Determinant is being studied.

4. PRELIMINARY LEMMAS

In order to prove the main result the following lemmas are required.

Lemma [2.1]

If $\omega \in \Omega$, for any complex number $\lambda, |\omega_2 - \lambda \omega_1^2| \leq \max\{1, |\lambda|\}$

Lemma [2.2]

If $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \Omega$

Then $|c_1^3 + c_3 + 2c_1 c_2| \leq 1$

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5. MAIN RESULT

Theorem 3.1: Let $f(z)$ be a function of the form (1.1) be in the class $S_{\mu,\beta}^{n,t}(q)$ then

$$|a_2| \le \frac{1}{[1+(1+\beta-\mu)t]^n}$$

$$|a_3| \le \frac{c_2}{[(2+\beta-\mu)t][1+(2+\beta-\mu)t]^n} + \frac{c_1^2}{2[2+\beta-\mu][1+(2+\beta-\mu)t]^n} + \frac{c_1^2}{[(1+\beta--\mu)t][2+\beta-\mu][1+(2+\beta-\mu)t]^n}$$

$$|a_4| \le \frac{1}{[(3+\beta-\mu)t][1+(3+\beta-\mu)t]}\left(1+\frac{1}{[(1+\beta-\mu)t][(2+\beta-\mu)t]}\right)$$

5.1 PROOF

Since the function defined by (1.1) belongs to the class $S_{\mu,\beta}^{n,t}(q)$,

$$\frac{D^{n+1}(\mu,\beta,t)f(z)}{D^n(\mu,\beta,t)f(z)} \prec \sqrt{1+\left(\omega(z)\right)^2}+\omega(z)=q(z) \qquad (3.1)$$

Thus,

$$D^{n+1}(\mu,\beta,t)f(z)-D^n(\mu,\beta,t)f(z)\omega(z)=D^n(\mu,\beta,t)f(z)\sqrt{1+\left(\omega(z)\right)^2} \qquad (3.2\ )$$

Where $\omega(0)=0, |\omega(z)|<1$ for $|z|<1$

From the definition in (1.2) we have

$$D^n(\mu,\beta,t)f(z) = z+[1+(1+\beta-\mu)t]^n a_2 z^2+[1+(2+\beta-\mu)t]^n a_3 z^3+[1+(3+\beta-\mu)t]^n a_4 z^4 + \cdots$$

$$D^{n+1}(\mu,\beta,t)f(z) = z+[1+(1+\beta-\mu)t]^{n+1} a_2 z^2+[1+(2+\beta-\mu)t]^{n+1} a_3 z^3 + [1+(3+\beta-\mu)t]^{n+1} a_4 z^4 + \cdots$$

$$D^n(\mu,\beta,t)f(z)\omega(z) = c_1 z^2+c_2 z^3+c_3 z^4+[1+(1+\beta-\mu)t]^n a_2 c_1 z^3+[1+(2+\beta-\mu)t]^n a_3 c_1 z^4+\cdots$$

$D^{n+1}(\mu,\beta,t)f(z)-D^n(\mu,\beta,t)f(z)\omega(z) = z+\{[1+(1+\beta-\mu)t]^{n+1}a_2-c_1\}z^2+\{[1+(2+\beta-\mu)t]^{n+1}a_3-\{[1+(1+\beta-\mu)t]^n a_2 c_1-c_2\}z^3+\{[1+(3+\beta-\mu)t]^{n+1}a_4-[1+(2+\beta-\mu)t]^n a_3 c_1\{[1+(1+\beta-\mu)t]^n a_2 c_2-c_3\}z^4+\ldots..$

On comparing the initial bounds of equation (3.3) and (3.4) we obtain.

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$$(i) a_2 = \frac{c_1}{[1+(1+\beta-\mu)t]^n} \tag{3.5}$$

$$(ii) a_3 = \frac{c_2}{[2+\beta-\mu)t]\cdot[1+(2+\beta-\mu)t]^n} + \frac{c_1^2}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}$$

$$+\frac{c_1^2}{[(1+\beta-\mu)t][(2+\beta-\mu)t][1+(2+\beta-\mu)t]^n}) \tag{3.6}$$

$$(iii) a_4 = \frac{1}{[(3+\beta-\mu)t][1+(3+\beta-\mu)t]}\left(\frac{c_1^3}{2[(1+\beta-\mu)t]}+\frac{c_1^3}{2[2+\beta-\mu)t]}\right.$$

$$\left.+\frac{c_1^3}{[(1+\beta-\mu)t][(2+\beta-\mu)t]}+\frac{c_1c_2}{[(1+\beta-\mu)t]}+\frac{c_1c_2}{[(2+\beta-\mu)t]}+c_1c_2+c_3\right) \tag{3.7}$$

It is known that the coefficients of the bounded function $\omega(z)$ satisfies the inequality that $|c_k| \leq 1$, so from (3.5),we have the first inequality that

$$|a_2| = \frac{1}{[1+(1+\beta-\mu)t]^n} \tag{3.8}$$

Also

$$[2+\beta-\mu)t][1+(2+\beta-\mu)t]^n|a_3| = \left|c_2+\frac{c_1^2}{2}+\frac{c_1^2}{(1+\beta-\mu)t}\right|$$

$$[2+\beta-\mu)t][1+(2+\beta-\mu)t]^n|a_3| = \left|c_2+\frac{c_1^2}{2}\left(\frac{(1+\beta-\mu)t-2}{2(1+\beta-\mu)t}\right)c_1^2\right| \tag{3.9}$$

Using the estimate that if $\omega(z)$ has the form(1.4) then

$$|c_2 - \lambda c_1{}^2| \leq \{\max_1 1, |\lambda|\}, for\ all\ \in \mathrm{C}$$

(3.10) from (3.9) and (3.10) we obtain

$$|\,[2+\beta-\mu)t]1+(2+\beta-\mu)t]^n a_3| \leq \frac{(1+\beta-\mu)t-2}{2(1+\beta-\mu)t}$$

Which gives the inequality that

$$|a_3| \leq \frac{(1+\beta-\mu)t-2}{2[(1+\beta-\mu)t][(2+\beta-\mu)t][1+(2+\beta-\mu)t]^n}$$

Also from (3.7) we obtain

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$$|[(3+\beta-\mu)t][1+(3+\beta-\mu)t]a_4| = \left|\frac{c_1^3}{2[(1+\beta-\mu)t]} + \frac{c_1^3}{2[2+\beta-\mu)t]} + \frac{c_1^3}{[(1+\beta-\mu)t][(2+\beta-\mu)t]}\right.$$

$$\left. + \frac{c_1c_2}{[(1+\beta-\mu)t]} + \frac{c_1c_2}{[(2+\beta-\mu)t]} + c_1c_2 + c_3\right|$$

$$= \left|\frac{[(2+\beta-\mu)t]+[(1+\beta-\mu)t]+2}{2[(1+\beta-\mu)t]}c_1^3\right.$$

$$\left. + \frac{[2+\beta-\mu)t][1+\beta-\mu)t]+[2+\beta-\mu)t]+[1+\beta-\mu)t]}{[2+\beta-\mu)t][1+\beta-\mu)t]}c_1c_2 + c_3\right|$$

$$= \left|\frac{[(2+\beta-\mu)t]+[(1+\beta-\mu)t]}{2[(1+\beta-\mu)t][(2+\beta-\mu)]}c_1^3 + \frac{2}{2[(1+\beta-\mu)t][(2+\beta-\mu)]}c_1^3\right.$$

$$\left. + \frac{[2+\beta-\mu)t][1+\beta-\mu)t]}{[2+\beta-\mu)t][1+\beta-\mu)t]}c_1c_2 + \frac{[2+\beta-\mu)t]+[1+\beta-\mu)t]}{[2+\beta-\mu)t][1+\beta-\mu)t]}c_1c_2 + c_3\right|$$

$$= \left|\frac{[(2+\beta-\mu)t]+[(1+\beta-\mu)t]}{2[(1+\beta-\mu)t][(2+\beta-\mu)t]}\left(c_1^3+2c_2+c_3\right)\right.$$

$$-\frac{2[(1+\beta-\mu)t][(2+\beta-\mu)t]-[(1+\beta-\mu)t]-[(2+\beta-\mu)t]}{2[(1+\beta-\mu)t][(2+\beta-\mu)t]}c_3$$

$$\left. + c_1\left(c_2 + \frac{c_1^2}{[1+\beta-\mu)t][2+\beta-\mu)t]}\right)\right|$$

Next we establish some properties of c_k and it is known that $p(z)$ is given by

$$\frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_zz^2 + \cdots = P(z)$$

defines the Caratheodory function with the property that $R_ep(z) > 0$ in U and that $|p_k| \leq 2(k = 1, 2, 3, \ldots)$ Equating the coefficient above we obtain

$$P_2 = 2(c_1{}^2 + c_2)$$

and

$$P_3 = 2(c_1{}^3 + 2c_1c_2 + c_3)$$

Hence

$$|c_1{}^2 + c_2| \leq 1$$

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and

$$|c_1{}^3+2c_1c_2+c_3| \leq 1$$

$$= \left| \frac{[(2+\beta-\mu)t]+[(1+\beta-\mu)t]}{2[(1+\beta-\mu)t][(2+\beta-\mu)t]} + \frac{2[(1+\beta-\mu)t][(2+\beta-\mu)t]-[(1+\beta-\mu)t]-[(2+\beta-\mu)t]}{2[(1+\beta-\mu)t][(2+\beta-\mu)t]} + \frac{1}{[1+\beta-\mu)t][2+\beta-\mu)t]} \right|$$

$$|[(3+\beta-\mu)t][1+(3+\beta-\mu)t]a_4| \leq 1 + \frac{1}{[1+\beta-\mu)t][2+\beta-\mu)t]}$$

$$|a_4| \leq \frac{1}{[(3+\beta-\mu)t][1+(3+\beta-\mu)t]} \left(1 + \frac{1}{[(1+\beta-\mu)t][2+\beta-\mu)t]}\right)$$

which complete the proof

THEOREM 3.2

If the function defined by (1.1) belong to the class $S_{\mu,\beta}^{n,t}(q)$

$$|a_3 - \lambda a_2^2| \leq$$

$$max\left(\frac{1}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}, \left|\lambda - \frac{[1+(+\beta-\mu)t]^{2n}[(1+\beta-\mu)t][2+(1+\beta-\mu)t]c_1^2}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}\right|\right)$$

$$(\lambda \in \mathbb{C})$$

Proof :

From *(3.6)* we have that

$$|a_3 - \lambda a_2^2| = \left| \frac{c_2}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} - \frac{1}{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2} \right.$$

$$\left. \left(\lambda - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2+[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}\}c_1^2\right)\right|$$

$$= \left| \frac{c_2}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} - \frac{1}{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2} \left(\lambda - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]\langle 2+[(1+\beta-\mu)t\rangle]}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}\right) c_1^2 \right|$$

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$$= \left| \frac{c_2}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} - \frac{1}{[1+(1+\beta-\mu)t]^{2k}[(1+\beta-\mu)t]^2} \right.$$

$$\left(\frac{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]\lambda}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} \right.$$

$$\left. \left. - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2 + [1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]}{2[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} \right) c_1^2 \right|$$

$$= \left| \frac{1}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} \left[c_2 - \frac{1}{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2} \right. \right.$$

$$\left. \left. \left(1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]\lambda - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t](2+(1+\beta-\mu)t]}{2} \right) c_1^2 \right] \right|$$

Applying Lemma [2.1], we obtain

$$= \frac{1}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} max\, [1, |[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]\lambda$$

$$\left. - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]^2 + [1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t]}{2} \right|]$$

$$= max \left[\frac{1}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]}, \left| \lambda - \frac{[1+(1+\beta-\mu)t]^{2n}[(1+\beta-\mu)t](2+(1+\beta-\mu)t]}{[1+(2+\beta-\mu)t]^n[(2+\beta-\mu)t]} \right| \right]$$

Theorem 3.3

If the function defined by (1.1) belong to the class$S_{\mu,\beta}^{n,t}(q)$,

Then,

$$|a_2a_4 - a_3^2| \le \frac{1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]} + \frac{1}{[(1+\beta-\mu)t]^2}$$

$$+ \frac{[2+\beta-\mu)t] + [(1+\beta-\mu)t] - [(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+ \frac{[2+\beta-\mu)t] + [(1+\beta-\mu)t] + 2 - 2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+ \frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}$$

Proof:

From equation (3.5), (3.6) and (3.7) where c_k are the coefficient of the Schwarz

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function,

Then,

$$a_2a_4 - a_3^2 =$$

$$\frac{c_1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}\left(\frac{[(2+\beta-\mu)t+(1+\beta-\mu)t+2]}{2[(1+\beta-\mu)t][(2+\beta-\mu)t]}c_1^3\right.$$

$$\left.+\frac{[(1+\beta-\mu)t][(2+\beta-\mu)t]+[(2+\beta-\mu)t+(1+\beta-\mu)t}{[(1+\beta-\mu)t][(2+\beta-\mu)t]}c_1c_2+c_3\right)$$

$$-\frac{1}{[(2+\beta-\mu)t]^2}\left(\frac{c_2}{[1+(2+\beta-\mu)t]^n}+\frac{[2+(1+\beta-\mu)t]}{2[(1+\beta-\mu)t]}c_1^2\right)$$

$$=\frac{c_1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}\left(c_1^3+2c_1c_2+c_3\right)$$

$$-\frac{1}{[(1+\beta-\mu)t]^2}\left(c_2+c_1^2\right)$$

$$-\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]-[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^2\left(c_2+c_1^2\right)$$

$$-\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]+2-2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^4$$

$$-\frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}c_1^4$$

Therefore,

$$|a_2a_4 - a_3^2|$$

$$=\left|\frac{c_1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}\left(c_1^3+2c_1c_2+c_3\right)\right.$$

$$-\frac{1}{[(1+\beta-\mu)t]^2}\left(c_2+c_1^2\right)$$

$$-\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]-[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^2\left(c_2+c_1^2\right)$$

$$-\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]+2-2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^4$$

$$\left.-\frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}c_1^4\right|$$

Applying Lemma [2.2], we obtain the result below

$$\leq\left|\frac{c_1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}\left(c_1^3+2c_1c_2+c_3\right)\right|$$

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$$+\left|\frac{1}{[(1+\beta-\mu)t]^2}(c_2+c_1^2)\right|$$

$$+\left|\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]-[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^2(c_2+c_1^2)\right|$$

$$+\left|\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]+2-2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}c_1^4\right|$$

$$+\left|\frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}c_1^4\right|$$

$$\leq\frac{1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}+\frac{1}{[(1+\beta-\mu)t]^2}$$

$$+\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]-[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]+2-2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+\frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}$$

Therefore we conclude that

$$|a_2a_4-a_3^2|=\frac{1}{[1+(1+\beta-\mu)t]^n[(1+\beta-\mu)t][1+(3+\beta-\mu)t]^n[(3+\beta-\mu)t]}+\frac{1}{[(1+\beta-\mu)t]^2}$$

$$+\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]-[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+\frac{[2+\beta-\mu)t]+[(1+\beta-\mu)t]+2-2[(2+\beta-\mu)t][(1+\beta-\mu)t}{[2(1+\beta-\mu)t][(2+\beta-\mu)t][1+(1+\beta-\mu)t]^n[1+\beta-\mu][1+(3+\beta-\mu)t]^n[3+\beta-\mu]t}$$

$$+\frac{2-(1+\beta-\mu)}{[2(1+\beta-\mu)t]^2[2+\beta-\mu]^2}$$

Which completes the proof.

6. CONCLUSION:

The Result Obtained in this work is the generalization of many known results previously studied by other authors.

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