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Analytical Modelling of Steady-State Solutions and its Implication

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ABSTRACT

The interaction between two biological species depends on the process of mathematical modelling. When two plant species compete for limited resources, we have used the analytical method to calculate four (4) steady-state solutions which represent valid coexistence two semi-trivial, and a trivial steady-state solution from a first order nonlinear differential equations with their biological implications. The full results of this study are presented and discussed.

Keywords: *Carrying Capacity, Analytic Method, Biological Species, Steady-State Solutions, Implication.*

1. INTRODUCTION

In the scenario of two competing plant species for limited resources, it is imperative to study the link between the carrying capacity on the coexistence of two interacting time dependent biological species. The analytic method in which the computational time can take a longer duration will be applied.

The carrying capacity of a biological species in an environment is the highest population size of the plant species that the surroundings can sustain in coexistence. The term of carrying capacity is not new within the mathematical literatures and ecological (Jin, Donovan, and Brettle, 2016; Ekaka-a, and Galadima, 2015; Damgaard, 2004; Kot, 2001; Meyer and Ausubel 1999). For instance, Meyer and Ausubel (1999) looked into carrying capacity: a model with logistically varying limit. They brought an extension to the widely-used logistic model of growth to a limit that in turn increases the carrying capacity. Using differential equation model, they also looked into the effect of this dynamic carrying capacity on the trajectories of simple growth models.

Uka and Ekaka-a (2012) considered using numerical simulation in the interaction of fish populations with bifurcation (coexistence steady-state solution model). Their findings was to provide short-term and a relatively long-term. There are other related works on steady-state solution (Leticia and Oleka 2016; Nafo, N.M. and Ekaka-a E.N. 2013; Pyragas, K., Pyragas, V., Kiss, Z., and Hudson, J.L. 2004; Cao, C., Ionnis, G., Kevrekdis and Titi E.S. 2001).

2. RESEARCH METHODOLOGY

2.1 Mathematical Formulations

For the purpose of this study, we have considered the following assumptions:

1. The growth of the two biological species over time is enhanced by their intrinsic growth rate values in the absence of the intra-competition and inter-competition.
2. The growth of the two biological species over time is enhanced by their intra-competition.
3. The growth of the two biological species over time is inhibited by their inter-competition.

We have considered the following continuous dynamical system of a nonlinear first order differential equation having the following mathematical structure, George (2018).

$$\frac{dx}{dt} = \alpha_1 x - \beta_1 x^2 - r_1 xy = F(x,y) \qquad (1)$$

$$\frac{dy}{dt} = \alpha_2 y - \beta_2 y^2 - r_2 xy = G(x,y) \qquad (2)$$

Having the initial data $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$.

We have assumed that the interaction focus on the independent variables $F(x, y)$ and $G(x, y)$ are continuous and partially differentiable.

We have also considered the following model specific parameter values estimated by Ekaka-a (2009),

$\alpha_1 = 0.168$, $\beta_1 = 0.0020339$, $r_1 = 0.0012$,

$\alpha_2 = 0.08$, $\beta_2 = 0.0018$, $r_2 = 0.0009$

For the purpose of understanding, the variables for these model equations are defined as follows;

$x(t)$ is called time dependent variable of x plant species.

$y(t)$ is called time dependent variable of y plant species.

α_1 is called the intrinsic growth rate for the x plant species (birth rate).

β_1 is called the intra-competition coefficient due to the interaction of the population x to inhibit the growth of x plant species.

α_2 is called the intrinsic growth rate for the y plant species (death rate).

β_2 is called the intra-competition coefficient due to the interaction of the population y to inhibit the growth of y plant species.

r_1 is called the inter competition coefficient due to the interaction of the population y to inhibit the growth of the x population.

r_2 is called the inter competition coefficient due to the interaction of the population x to inhibit the growth of the y population.

2.2 Method of Analysis: Determination of Steady-State Solutions

From (1) and (2);

This can be rewritten as

$$\frac{dx}{dt} = x(\alpha_1 - \beta_1 x - r_1 y) \qquad (3)$$

$$\frac{dy}{dt} = y(\alpha_2 - \beta_2 y - r_2 x) \qquad (4)$$

At a steady state solution

$$\frac{dx}{dt} = 0$$

and

$$\frac{dy}{dt} = 0$$

$$x(\alpha_1 - \beta_1 x - r_1 y) = 0 \qquad (5)$$

$$y(\alpha_2 - \beta_2 y - r_2 x) = 0 \qquad (6)$$

$$\alpha_1 - \beta_1 x - r_1 y = 0$$

$$\alpha_2 - \beta_2 y - r_2 x = 0$$

$$\beta_1 x + r_1 y = \alpha_1 \qquad (7)$$

$$r_2 x + \beta_2 y = \alpha_2 \qquad (8)$$

Using Crammers Rule;

1. When $x \neq 0$, $y \neq 0$

   From (7) and (8) we have;

$$\begin{pmatrix} \beta_1 & r_1 \\ r_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Suppose $A = \begin{pmatrix} \beta_1 & r_1 \\ r_2 & \beta_2 \end{pmatrix}$

$$\det A = |A| = \begin{vmatrix} \beta_1 & r_1 \\ r_2 & \beta_2 \end{vmatrix} = \beta_1\beta_2 - r_1 r_2$$

$$\Delta A_1 = \begin{pmatrix} \alpha_1 & r_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

$$|\Delta A_1| = \begin{vmatrix} \alpha_1 & r_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - r_1\alpha_2$$

$$\Delta A_2 = \begin{pmatrix} \beta_1 & \alpha_1 \\ r_2 & \alpha_2 \end{pmatrix}$$

$$|\Delta A_2| = \begin{vmatrix} \beta_1 & \alpha_1 \\ r_2 & \alpha_2 \end{vmatrix} = \beta_1\alpha_2 - \alpha_1 r_2$$

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Therefore, $x = \frac{|\Delta A_1|}{|A|}$

$$x = \frac{\alpha_1\beta_2 - r_1\alpha_2}{\beta_1\beta_2 - r_1r_2}$$

$$y = \frac{|\Delta A_2|}{|A|}$$

$$= \frac{\beta_1\alpha_2 - \alpha_1 r_2}{\beta_1\beta_2 - r_1r_2}$$

Therefore, $(x, y) = \left(\frac{\alpha_1\beta_2 - r_1\alpha_2}{\beta_1\beta_2 - r_1r_2}, \frac{\beta_1\alpha_2 - \alpha_1 r_2}{\beta_1\beta_2 - r_1r_2}\right)$ is a unique positive steady-state solution provided;

1. $\beta_1\beta_2 > r_1r_2$
2. $\alpha_1\beta_2 > r_1\alpha_2$
3. $\beta_1\alpha_2 > \alpha_1 r_2$

3. RESULTS: IMPLICATIONS OF STEADY-STATE SOLUTIONS

In this section, we shall state the implication of the steady-state solutions;

1. The steady-state solution (0, 0) represents the extinction of the two biological species.
2. The steady-state solution $(\frac{\alpha_1}{\beta_1}, 0)$ represents the extinction of the second biological species at the instance when the first biological species survival at its carrying capacity value of $\frac{\alpha_1}{\beta_1}$.
3. The steady-state solution $(0, \frac{\alpha_2}{\beta_2})$ represents the extinction of first biological species at the instance when the second biological species survival at its carrying capacity value of $\frac{\alpha_2}{\beta_2}$. By implication, the first biological species will survival at its carrying capacity value of 82.60 approximately whereas the second biological species will survival at its carrying capacity values of 44.44 approximately.

4. CONCLUSION

In the context of two competing biological species, we have used a mathematical method to calculate a valid unique positive steady-state solution which represents the concept of coexistence from which a trivial steady-state solution representing the ecological risk of extinction and two other semi-trivial steady-state solutions were derived that represent the boundaries and the magnitudes of survival of two plant species having tow dis-similar carrying capacity values of 82.60 and 44.44 approximately. A future research work will be to investigate how the variation of a model parameter value one-at-a-time will affect the boundary of coexistence.

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