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MATHEMATICAL MODELING OF CARDIAC BLOOD FLOW IN HUMANS

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Abstract:

Lumped parameter model is a very useful type of mathematical modelling where the physical system is made analogous to an electrical network. Lumped parameter model is represented graphically by a circuit diagram in which vertices represent the voltages and the edges the current in the circuit. The mathematical analysis of such a circuit model is very convenient and eases the analysis of the actual physical system. These models are constructed by building the analogy between cardiovascular system and electrical circuit.This analogy makes use of simple ordinary differential equations in time which can be solved either numerically or analytically.

A mathematical model to model the blood pressure variation in the four chambers of heart at representative altitudes of h= 2 km, h = 4 km and h = 5 kmusing lumped compartments of blood circulation is presented in this paper. This lumped parameter model consists of eight compartments that include the pumping heart, the systemic circulation and the pulmonary circulation. The governing equations for pressure and flow in each compartment are derived from the following three equations: Ohm's law, conservation of volume and the definition of compliances. The g-factor is added to the model in order to accommodate for the effects due to gravitation.

Keywords: Cardiovascular system, lumped parameters, Electrical analogy, blood vessels, cardiac phases, g-factor.

INTRODUCTION

The lumped parameter electric model proposed by Eunok Jung and Wanho Lee [1] has been studied extensively and in the present work, the equations have been modified by considering the gravity factor along with the definition of step function in order to study the variation of blood pressure inside the eight compartments during one typical cardiac cycle at higher altitudes.

In this model the cardiac cycle is represented as alumped pulsatile model by describing a network of the compliance and resistance vessels.

Electrical Analogy:

Cardio vascular system≈ Electric circuit

Blood pressure ≈ Voltage

Blood flow≈ Current

Volume of blood ≈ Charge

Blood vessel resistance≈ Resistance

Compliance≈ Capacitance

Valves≈ Diodes

This model relatesthe blood flow and pressure using relatively simple ordinary differential equations in time.In order to derive a system of differential equations for blood circulation, the following three principles from physiology and physics are used:

Ohm's law : $Q(t) = \frac{\Delta P(t)}{R}$ --(1)

definition of compliance : $\frac{dV(t)}{dt} = Q_{in}(t) - Q_{out}(t)$ --------------------------------------(2)

volume conservation : $V(t) = V_d + C(t)\ P(t)$ ---(3)

The modelling of cardiac valves is done using step function S(t), and the valves represented as $S_{Mi}(t)$ for mitral valve, $S_{Tr}(t)$ for tricuspid valve, $S_{Ao}(t)$ for pulmonary valve and $S_{Pu}(t)$ for pulmonary valve and defined as follows [2]:

$$S_{Mi}(t)=\begin{cases}1 & if \quad P_{LA}\geq P_{LV}\\ 0 & if \quad P_{LA}<P_{LV}\end{cases} \quad S_{Tr}(t)=\begin{cases}1 & if \quad P_{RA}\geq P_{RV}\\ 0 & if \quad P_{RA}<P_{RV}\end{cases}$$

$$S_{Ao}(t)=\begin{cases}1 & if \quad P_{LV}\geq P_{sa}\\ 0 & if \quad P_{LV}<P_{sa}\end{cases} \quad S_{Pu}(t)=\begin{cases}1 & if \quad P_{RV}\geq P_{pa}\\ 0 & if \quad P_{RV}<P_{pa}\end{cases}$$

Using the equations (1), (2), (3) and the values of the definition, the following eight differential equations can be derived to model one cardiac cycle [1, 2, 3, 4].

$$\frac{d(C_{LA}(t)P_{LA}(t))}{dt}=\frac{P_{pv}(t)-P_{LA}(t)}{R_{pv}}-\frac{S_{Mi}(t)(P_{LA}(t)-P_{LV}(t)}{R_{Mi}} \text{------------------------------------(4)}$$

$$\frac{d(C_{LV}(t)P_{LV}(t))}{dt}=\frac{S_{Mi}(t)(P_{LA}(t)-P_{LV}(t))}{R_{Mi}}-\frac{S_{Ao}(t)(P_{LV}(t)-P_{sa}(t))}{R_{Ao}} \text{--------------------------(5)}$$

$$\frac{C_{sa}dP_{sa}(t)}{dt}=\frac{S_{Ao}(t)(P_{LV}(t)-P_{sa}(t))}{R_{Ao}}-\frac{P_{sa}(t)-P_{sv}(t)}{R_{s}} \text{--(6)}$$

$$\frac{C_{sv}dP_{sv}(t)}{dt}=\frac{P_{sa}(t)-P_{sv}(t)}{R_{s}}-\frac{P_{sv}(t)-P_{RA}(t)}{R_{sv}} \text{--(7)}$$

$$\frac{d(C_{RA}(t)P_{RA}(t))}{dt}=\frac{P_{sv}(t)-P_{RA}(t)}{R_{sv}}-\frac{S_{Tr}(t)(P_{RA}(t)-P_{RV}(t)}{R_{Tr}} \text{------------------------------------(8)}$$

$$\frac{d(C_{RV}(t)P_{RV}(t))}{dt}=\frac{S_{Tr}(t)(P_{RA}(t)-P_{RV}(t))}{R_{Tr}}-\frac{S_{Pu}(t)(P_{RV}(t)-P_{pa}(t))}{R_{Pu}} \text{-----------------------(9)}$$

$$\frac{C_{pa}dP_{pa}(t)}{dt}=\frac{S_{Pu}(t)(P_{RV}(t)-P_{pa}(t))}{R_{Pu}}-\frac{P_{pa}(t)-P_{pv}(t)}{R_{p}} \text{---------------------------------------(10)}$$

$$\frac{C_{pv}dP_{pv}(t)}{dt}=\frac{P_{pa}(t)-P_{pv}(t)}{R_{p}}-\frac{P_{pv}(t)-P_{LA}(t)}{R_{pv}} \text{--(11)}$$

At sea level, the atmospheric pressure is about 1.03 kg/cm^2 allowing oxygen to easily pass through the vascular membrane. However, at higher altitudes, the lower air pressure makes it difficult for oxygen to pass through the vascular membrane, which leads to a medical condition called *hypoxia* - deprivation of oxygen level in the blood. At higher altitudes, as the consequence of acute hypoxia there is an increase in heart rate and myocardial contractivity.

It is observed that there is quite a large increase in the blood pressure in all compartments. The initial response to the altitude is observed to be the change in the blood pressure.

From literature it is understood that the effects of hypoxia are evident at an altitude of 2 km above sea level and life sustenance is difficult after an altitude of 5 km above sea level for a normal healthy human being. Hence, in this study the variation of blood pressure in heart chambers at altitudes of h = 2 km and h = 5 km have been considered. In addition, a study of variation of blood pressure at an altitude h = 4 km has also been undertaken to see the variation in between these two extremes [5, 6].

The gravitational factor appears to play an important role in blood circulation during cardiac cycle as there is a lot of change in atmospheric pressure and blood pressure at high altitudes. Hence, the g-factor is introduced into the model and is defined as follows.

$g_h\left(\frac{(d\Delta t)^2}{10^6}\right)$ where $g_h = g_0\left(\frac{r_e}{r_e+h}\right)^2$; Δt refers to time interval and 'd' refers to the diameter of the valve; r_e(mean radius of earth) = 6371km, g_o = 9.81m/s^2 and h, the altitude considered in km. The diameters of valves considered for this work are as shown in Table 1.

Table 1: Mean Diameter values of valves

sv (diameter, mm)	**Pv (diameter, mm)**
Superior venacava 20 Inferior venacave20	Right superior 11.9 Right inferior 12.7 Left superior 10.05 Left inferior 9.45
Tricuspid valve (diameter, mm) 28	**Mitral valve (diameter, mm) 24**
Pa (diameter, mm) 30	**sa (diameter, mm) Aorta 30**

The g factor value of each of pulmonary vein, systemic vein, pulmonary artery, systemic artery, mitral valve and tricuspid valve at altitudes h = 2 km, 4 km and 5 km is added to the differential equations (4) - (11) to model the circulatory system to study the variation of pressure in eight compartments of the lumped parameter model

Based on literature [1], the physiological resting values of constant compliances, resistances and dead volumes at each compartment as shown in Table 2, the calculations are done during one cardiac cycle at altitudes h = 2km, 4km and 5 km.

Table 2: Initial values of pressure (mm/Hg), resistance (mmHg/liter/min), capacitance (liter/mmHg) and value of dead volume (liter) of eight compartments

Compartment	Initial value of pressure (mmHg)	Compartment	Initial value of resistance (mmHg/liter /min)	Compartment	Initial value of Capacitance (liter/mmHg)	Compartment	Value of the dead Volume (liter)
LA	10	**Mi**	0.01	**Sa**	0.0018	**LA**	0.03
LV	9.8	**Sa**	16.964	**Sv**	0.7	**LV**	0.03
Sa	100	**Ao**	0.01	**Pa**	0.0046	**Sa**	0.825
Sv	5	**Sv**	0.05	**Pv**	0.04	**Sv**	0
RA	5	**Tr**	0.01			**RA**	0.03
RV	4.8	**Pa**	1.786			**RV**	0.03
Pa	20	**Pu**	0.01			**Pa**	0.038
Pv	10	**Pv**	0.05			**Pv**	0

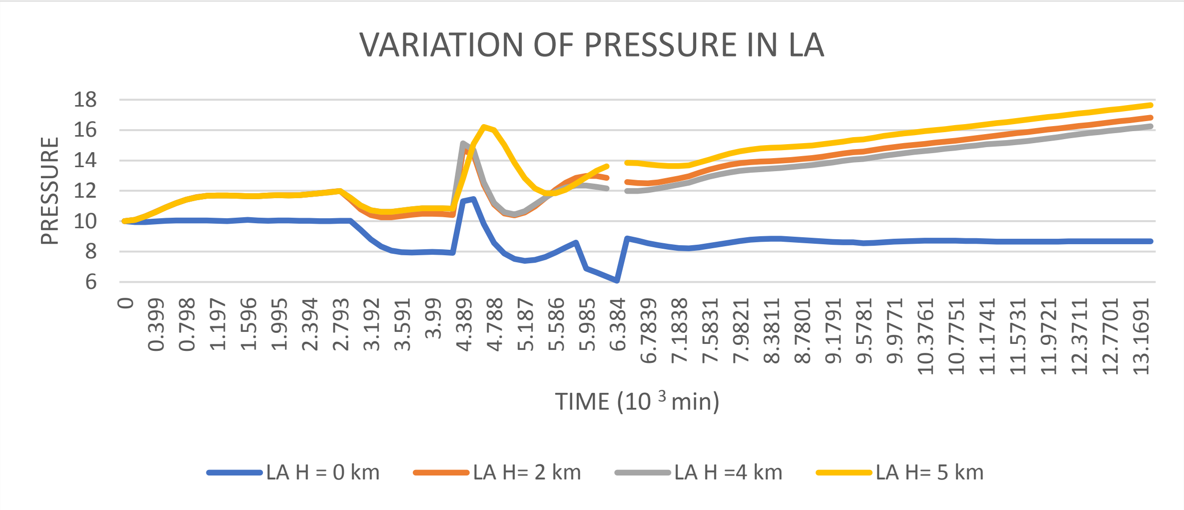
Table 3 gives the values of the step function for each of the step function for different phases of cardiac cycle.

Table 3: Values of step functions during seven phases

Phase	**Phase description**	**Observations**	**Values of Step function**
I	Atrial contraction	$P_{LA}>P_{LV}$, $P_{LV}<P_{sa}$,$P_{RA}>P_{RV}$ and $P_{RV}<P_{pa}$	$S_{Mi}=1$,$S_{Ao}=0$, $S_{Tr}=1$ and $S_{Pu}=0$
II	Isovolumetric contraction	$P_{LA}<P_{LV}$, $P_{LV}>P_{sa}$, $P_{RA}<P_{RV}$ and $P_{RV}>P_{pa}$	$S_{Mi}=0$, $S_{Ao}=1$, $S_{Tr}=0$ and $S_{Pu}=1$
III	Rapid ejection	$P_{LA}<P_{LV}$, $P_{LV}>P_{sa}$, $P_{RA}<P_{RV}$ and$P_{RV}>P_{pa}$	$S_{Mi}=0$, $S_{Ao}=1$, $S_{Tr}=0$ and $S_{Pu}=1$
IV	Reduced ejection	$P_{LA}=P_{LV}$, $P_{LV}<P_{sa}$,, $P_{RA}=P_{RV}$ and $P_{RV}<P_{pa}$	$S_{Mi}=1$, $S_{Ao}=0$, $S_{Tr}=1$ and $S_{Pu}=0$
V	Isovolumetric relaxation	$P_{LA}=P_{LV}$, $P_{LV}<P_{sa}$, $P_{RA}=P_{RV}$ and $P_{RV}<P_{pa}$	$S_{Mi}=1$, $S_{Ao}=0$, $S_{Tr}=1$ and $S_{Pu}=0$
VI	Rapid filling	$P_{LA}>P_{LV}$, $P_{LV}<P_{sa}$, $P_{RA}>P_{RV}$ and $P_{RV}<P_{pa}$	$S_{Mi}=1$, $S_{Ao}=0$, $S_{Tr}=1$ and $S_{Pu}=0$
VII	Reduced filling	$P_{LA}>P_{LV}$, $P_{LV}<P_{sa}$,, $P_{RA}>P_{RV}$ and $P_{RV}<P_{pa}$	$S_{Mi}=1$, $S_{Ao}=0$, $S_{Tr}=1$ and $S_{Pu}=0$

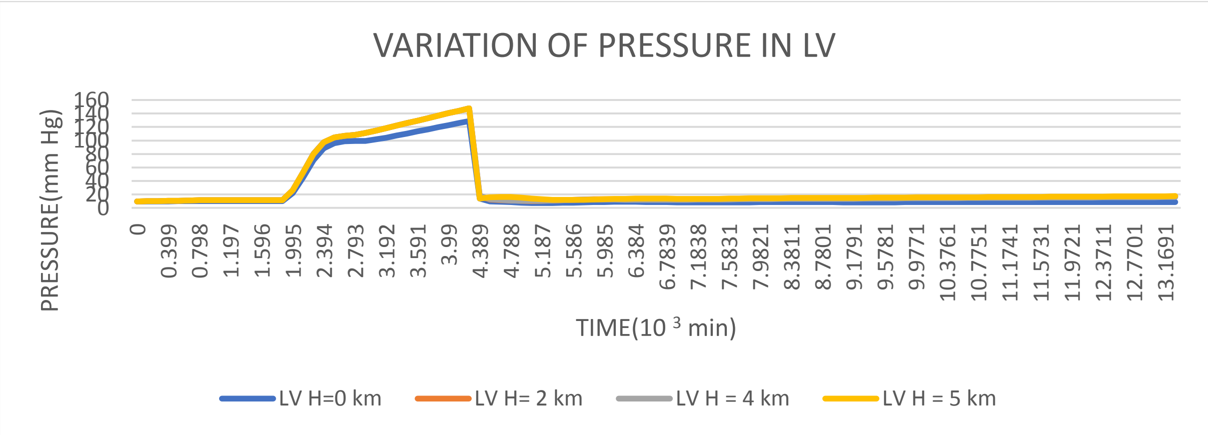
Taking these values from Table 2 as initial values for pressure, compliance and volume and choosing the values of the step function for different phases of cardiac cycle from Table 3 and making use of the algorithm of backward Euler's method, the calculations are done for each of the phases. The variation of blood pressure in four chambers of heart during one cardiac cycle at sea level and at altitudes of 2 km, 4 km and 5 km above sea during seven phases are tabulated below.

Variation of pressure at h = 0 km, h=2 km, h=4 km,and h= 5 km in the Left Atrium at the end of one cardiac cycle, i.e., at the end of each of the seven phases.



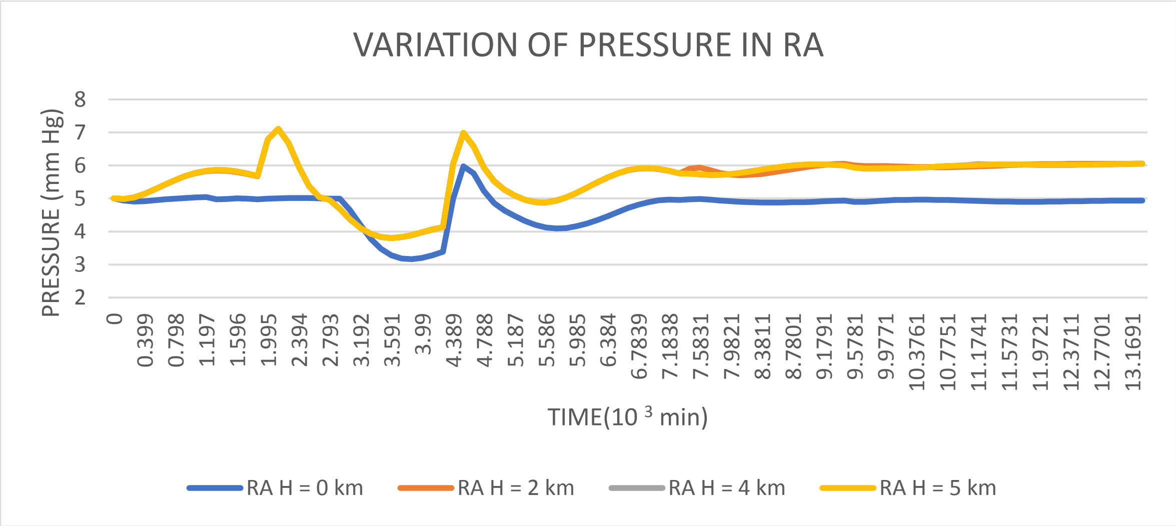
Graph 1: Variation of blood pressure in left atrium during one cardiac cycle at sea level and at altitudes of 2 km, 4 km and 5 km above sea level

Variation of pressure at h = 0 km, h = 2 km, h = 4 km, and h = 5 km in the Left Ventricle at the end of one cardiac cycle, i.e., at the end of each of the seven phases.



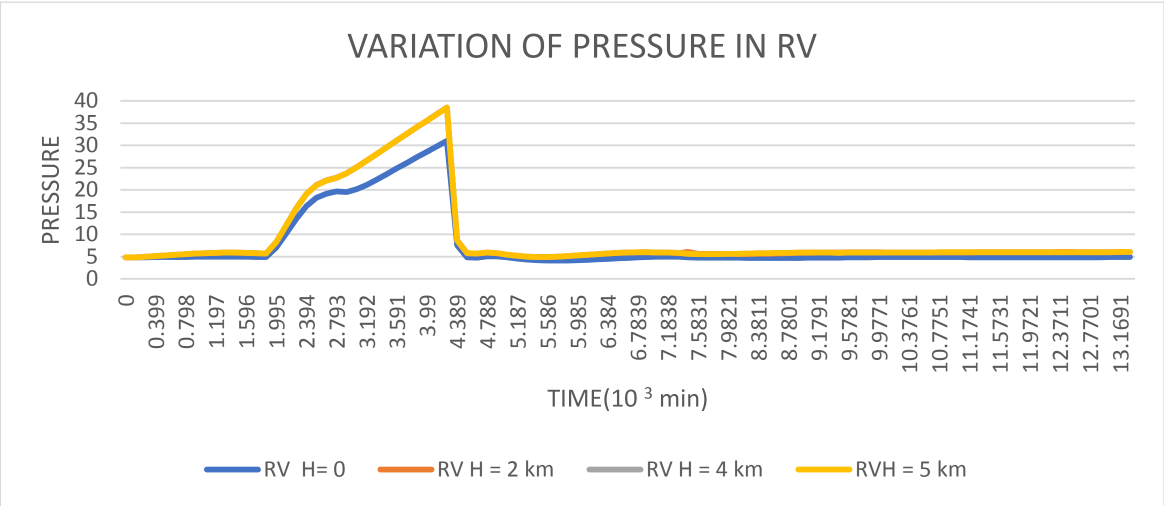
Graph 2: Variation of blood pressure in left ventricle during one cardiac cycle at sea level and at altitudes of 2 km, 4 km and 5 km above sea level

Variation of pressure at h = 0 km, h = 2 km, h = 4 km, and h = 5 km in the Right Atrium at the end of one cardiac cycle, i.e., at the end of each of the seven phases.



Graph 3: Variation of blood pressure in right atrium during one cardiac cycle at sea level and at altitudes of 2 km, 4 km and 5 km above sea level

Variation of pressure at h = 0 km, h = 2 km, h = 4 km, and h = 5 km in the RightVentricle at the end of one cardiac cycle, i.e., at the end of each of the seven phases



Graph 4: Variation of blood pressure in right atrium during one cardiac cycle at sea level and at altitudes of 2 km, 4 km and 5 km above sea level

Studying graphs 1 to 4,it is observedthat, there is a steep increase in blood pressure in left atrium at higher altitudes validating the literature result asserting that at high altitudes, due to hypoxia, the left ventricle isn't able to pump out enough of the blood it receives from the lungs, as a result the pressure inside the left atrium increases considerably. Also, there is an increase in pressure in both atria and ventricles. However, there is larger increase in atrial pressure than in ventricular pressure.

According to literature, at sea level on an average for a normal healthy person, the systolic blood pressure is around 120 mm Hg and diastolic blood pressure 80 mm Hg. At higher altitudes, the systolic pressures obtained were observed around 147.8 mm Hg and diastolic pressure values around 108.3 mm Hg. There is a prominent increase ofpressure in pulmonary artery and pulmonary veins. The elevation in pressure explains the effects of hypoxia.

EFFECTS:

The blood pressure is chronically elevated as we move towards higher altitudes [7, 8, 9]. At higher altitudes, the possible symptoms one may experience are dizziness, headache, blurred vision, tinnitus(ringing sound in the ears), nausea and vomiting.

If proper precaution is not taken to reduce the blood pressure as we move towards higher altitudes, this untreated high blood pressure can damage organs such as heart, kidneys or eyes.

Thus the primary concern is to take proper treatment to lower the blood pressure to normal level through appropriate combination of drugs that achieves this goal.

CONCLUSION:

- The results obtained indicate the effects due to variation of blood pressure with respect to change in altitudes.
- It is observed that there is a steep increase in blood pressure in left atrium as one moves to higher altitudes. The reason being that the left ventricle isn't able to pump out enough of the blood it receives from the lungs; as a result the pressure inside the left atrium increases considerably.
- Also, there is an increase in blood pressure in both atria and ventricles. However, there is a larger increase in atrial pressure than in ventricular pressure.
- The use of the equations described in this work is evident in all the eight compartments as described in the developed mathematical model at various altitudes. Thus the proposed mathematical modelling remains valid within the acceptable range.
- However, there is a large scope for improvement in the model by considering the effects of the temperature and viscosity on the variation of blood pressure.

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