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Comprehensive Study on Investigation on Topology of the Line and Plane

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ABSTRACT

In this paper we have discussed the definitions and the basic properties of open sets, closed sets, accumulation points or limit points and sequence. Sets may be neither open nor closed. The reader should not confuse the concept "limit point of a set" with the different, though related, concept "limit of a sequence". Some of the solved and supplementary problems will show the relationship between these two concepts. Observe that $\langle a_n : n \in N \rangle$ *denotes a sequence and is a function. On the other hand,* $\{a_n : n \in N\}$*denotes the range of the sequence and is a set. We have given several characterizations of these sets. We discuss the definitions of bounded sequence, convergent sequence, Cauchy sequence and their relations. Then we can study every bounded sequences of real numbers contains a convergent subsequence and every Cauchy sequence of real numbers converges to a real number. Let A be a bounded, infinite set of real numbers. Then A has at least one accumulation point. We express the definition of topology, usual topology and topological space. The creation of topology the science of spaces and figures that remains unchanged under continuous deformations represents a phenomenon of this kind, but of a distinctly modern variety. Then we begin our study of some properties topological spaces by making the idea of being connected that is being in one piece. We observe that, for the usual topology on the line R and in the plane* R^2*. Finally we express the characterizations of the discrete topological space and indiscrete topological space.*

Key Words: Sets, Sequences, Topology, Topological Spaces.

1. INTRODUCTION

This paper provides the discussion in three sections. The first one is the basis definitions of open sets and their properties on topology on the line R and in the plane R^2. The second is the basis definitions of closed sets and their properties on topology on the line R and in the plane R^2. And finally, we study the basis properties of topological spaces.

The set of real numbers, denotes by R, plays a dominant role in mathematics and in particular, in analysis. In fact, many concepts in topology are abstractions of properties of set of real numbers. We assume the reader is familiar with the geometric representation of R by means of the points on a straight line and in the plane, by means of the points on a plane [1].

2. BASIS PROPERTIES

2.1 Open Sets

Let A be a subset of R. A point $p \in A$ is an interior point of A if and only if $p \in some\ open\ interval\ S_p$ which is contained in A; $p \in S_p \subset A$. The set $A$is open if and only if each of its points is an interior point [1].

2.1.1 Example

(i) An open interval $A = (a, b)$ is an open set. For, we may choose $S_p = A$ for each $p \in A$.

(ii) The real line R, itself, is open since any open interval S_p must be a subset of R, that is $p \in S_p \subset R$.

(iii) The closed interval $B = [a, b]$ is not an open set.
For, any open interval containing $a\ or\ b$ must contain points outside of B. Hence the end points $a\ and\ b$ are not interior points of B.

(iv) The empty set $\emptyset$ is open since there is no point in $\emptyset$ which is not an interior point.

(v) The infinite open intervals, i.e., the subsets of R defined and denoted by
$(a, \infty) = \{x | x \in R, x > a\}$, $(-\infty, a) = \{x | x \in R, x < a\}\ and\ (-\infty, \infty) = \{x | x \in R\} = R$ are open sets.
On the otherhand, the infinite closed intervals, i.e., the subsets of R defined and denoted by $[a, \infty) = \{x | x \in R, x \geq a\}$, $(-\infty, a\,] = \{x | x \in R, x \leq a\}$ are not open sets since $a \in R$ is not an interior point of either $[a, \infty)\ or\ (-\infty, a]$.

2.1.2 Theorem

The union of any number of open sets in R is open.

2.1.3 Theorem

The intersection of any finite number of open sets in R is open.

Next, we will show that an arbitrary intersection of open sets need not be open.

2.1.4 Example

Let $A_n = \left\{\left(-\frac{1}{n}, \frac{1}{n}\right) \middle| n \in N\right\}$ be the class of open intervals. Show that $\bigcap_{n=1}^{\infty} A_n$ is not an open set.

Solution

$A_n = \left\{\left(-\frac{1}{n}, \frac{1}{n}\right) \middle| n \in N\right\}$, i.e., $\left\{(-1,1), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{3}, \frac{1}{3}\right), \dots\right\}$.

The intersection $\bigcap_{n=1}^{\infty} A_n = \{0\}$ of the open intervals consists of the single point 0 which is not an open set.

Hence an arbitrary intersection of open sets is not an open set.

2.2 Open Disc

A open disc D in the plane R^2 is the set of points inside a circle, with center $p = \langle a_1, a_2 \rangle$ and radius > 0 , i.e., $D = \{q \in R^2 : d(p,q) < \delta\}$, where $d(p,q)$ denotes the usual distance between two points $p = \langle a_1, a_2 \rangle$ and $q = \langle b_1, b_2 \rangle$ in R^2.

Let A be a subset of R^2. A point plane $p \in A$ is an interior point of A if and only if p belongs to some open disc D_p which is contained in $A: p \in D_p \subset A$.

The set A is open if and only if each of its points is an interior point [1-2].

2.2.1 Example

Show that the union of any number of open subsets of R^2 is open.

Solution

Let A be a class of open subsets of R^2.

Let $H = \bigcup\{G : G \in A\}$.

Let $p \in H$.

We must show that p is an interior point of H.

Since $p \in H, \exists G_0 \in A: p \in G_{0.}$

But G_0 is an open set, hence there exists an open disc D_p such that $p \in D_p \subset G_0. G_0 \subset H \; and \; so \; D_p \subset H. i.e. p \in D_p \subset H.$

Hence p is an interior point of H and so $H = UG$ is open set.

2.2.2 Example

Show that the intersection of any finite number of open subsets of R^2 is open.

Solution

Let G and H be open subsets of R^2.

To prove, $G \cap H$ is open.(i.e., $p \in S_p \subset G \bigcap H$).

Let $p \in G \bigcap H$.

Then $p \in G$ and $p \in H$.

Since G and H are open sets, there exists open disc D_1 and D_2 such that $p \in D_1 \subset G$ and $p \in D_2 \subset H$.

Then $p \in D_1 \cap D_2 \subset G \cap H$.

But $D_1 \cap D_2 = D$.

Therefore p is an interior point of $G \cap H$.

Hence $G \cap H$ is open.

2.2.3 Example

Prove that every open subset G of the plane R^2 is the union of open discs.

Solution

Since G is open, for each $p \in G$ there is open disc D_p such that $p \in D_p \subset G$.

Then $G = \cup \{D_p | p \in G\}$.

2.2.4 Example

Let G be an open subset of R^2, $p \in G$. Prove that there exists an open disc D with center p such that $p \in D \subset G$.

Solution

By definition of an interior point, there exists an open disc, $D_1 = \{q \in R^2 : d(p_1, q) < \delta\}$ with center p_1 and radius δ such that $p \in D_1 \subset G$. So $d(p_1, p) < \delta$.

Let $r = \delta - d(p_1, p) > 0$ and $D = \left\{q \in R^2 : d(p,q) < \frac{r}{2}\right\}$.

Then $p \in D \subset D_1 \subset G$, where D is an open disc with center p.

2.3 Accumulation Point or Limit Point

Let A be a subset of R. A point $p \in R$ is an accumulation point or limit point of A if and only if every open set G containing p

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contains a point of A different from p ; i.e., G open, $p \in G$ implies $A \cap (G \backslash \{p\}) \neq \emptyset$.

The set of accumulation points of A, denoted by A', is called the derived set of A.[1-3].

2.3.1 Example

(i) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

The point "0" is an accumulation point of A since every open set G with $0 \in G$ contains an open invertal $(-a_1, a_2) \subset G$ with -$a_1 < 0 < a_2$ which contains points of A. The limits point 0 of A dose not belong to A and A does not contain any other limit points. Hence $A' = \{0\}$.

(ii) Let Q be the set of rational numbers.

Every real number $p \in R$ is a limit point of Q since every open set contains rational numbers.

Hence $Q' = R$.

(iii) The set of integers $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Then Z does not have any points of accumulation. Hence $Z' = \emptyset$.

2.3.2 Example

Determine the accumulation points of each set of real numbers: (i) N, (ii) $(a, b]$, (iii) Q^c, *the set of irrational points*.

Solution

(i) $N =$ the sets of natural numbers. N does not have any limit points.

For, if "a" is any real number, we can find $\delta > 0$ such that open set (a-$\delta, a + \delta$) contains no point of N other than "a".

Hence $N' = \emptyset$.

(ii) (a,b].

Every point $p \in [a, b]$ is a limit point of (a,b].

Since every open interval containing $p \in [a, b]$ will contain points of (a,b] other than p.

Hence $(a, b]' = [a, b]$.

(iii) Q^c, *the set of irrational points*.

Every real number $p \in R$ is a limit point of Q^c since every open interval containing $p \in R$

Will contain points of Q^c other than p.

Hence $(Q^c)' = R$.

2.3.3 Example

Let A' be the derived set, i.e., the set of limit points of a set A. Find sets A such that

(i) A and A' are disjoint,

(ii) A is a proper subset of A',

(iii) A' is a proper subset of A,

(iv) $A' = A$.

Solution

(i) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

Then $A' = \{0\}$ and so A and A' are disjoint.

(ii) Let $A = (a, b]$.

Then $A' = [a, b]$ and so $A \subset A', A \neq A'$.

Hence A is a proper subset of A'.

(iii) Let $A = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

Then $A' = \{0\}$ and so $A \subset A', A \neq A'$.

Hence A' is a proper subset of A.

(iv) Let $A = [a, b]$.

Then each point of A is a limit of A.

So $A' = [a, b]$.

Hence $A = A'$.

2.4 Closed Sets

A subset A of R is a closed set if and only if its complement A^c is an open set [4].

2.4.1 Theorem

A subset A of R is a closed if and only if A contains each of its points of accumulation.

2.4.2 Example

(i) The closed interval [a,b] is a closed set since its complement $(-\infty, a) \cup (b, \infty)$, the union of two open infinite intervals, is open.

(ii) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is not closed since 0 is a limit point of A but $0 \notin A$.

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(iii) The empty set $\emptyset$ and the entire line R are closed sets since their complements R and $\emptyset$, respectively, are open sets.

2.4.3 Example

A set F is closed if and only if its complement F^c is open.

Solution

Since $(F^c)^c = F$; So F is the complement of F^c. Thus, by definition, F is closed if and only if F^c is open.

2.4.4 Example

Prove that the union of a finite number of closed sets is closed.

Proof

Let $F_{1,\ldots,}F_m$ be closed sets and $F = F_1 \cup \ldots \cup F_{m.}$

Then $F = (F_1 \cup \ldots \cup F_m)^c$

$= {F_1}^c \cap \ldots \cap {F_m}^c$.

F_i is closed and ${F_i}^c$ is open.

So F^c is the intersection of a finite number of open sets ${F_i}^c$.

Thus F^c is also open.

Hence $(F^c)^c = F$ is closed.

2.4.5 Example

Proved that the intersection of any number of closed sets is closed.

Solution

Let $\{F_i\}$ be a class of closed sets and $F = \bigcap_i F_i$.

Then $F^c = (\bigcap_i F_i)^c$

$F^c = \bigcup_i {F_i}^c$.

F_i is closed and ${F_i}^c$is open.

So F^c is the union of any number of open sets ${F_i}^c$.

Thus F^c is open.

Hence $(F^c)^c = F$ is closed.

2.4.6 Example

Proved that a subset of R^2 is closed if and only if it contains each of its accumulation points.

Solution

Suppose $p$is a limit point of a closed set $F \subset R^2$. Then every open disc containing p contains points of F other than p.

Hence there cannot be open disc D_p containing p which is completely contained in F^c.

So p is not an interior point of F^c. But F^c is open, since F is closed.

Then $p \notin F^c, p \in F$.

Conversely, suppose F contains each of its accumulation points.

Let $p \in F^c$.

Then p is not a limit point of F.

Hence there exists at least one open disc D_p containing p such that D_p does not contain any points of F.

So $D_p \subset F^c$.

Hence p is an interior point of F^c and F^c is open. Then F is closed.

2.5 Sequence

A sequence denoted by $\langle S_n \rangle = \langle s_1, s_2, s_3, \ldots, s_{n,} \ldots \rangle, n \in N$ is a function whose domain is $N = \{1,2,\ldots\}$.

The image $S_n, n \in N$ is called the n^{th} term of the sequence. [4]

2.6 Bounded

A sequence $\langle S_n \rangle$ is said to be bounded if its range $\{S_n : n \in N\}$ is a bounded set.

2.6.1 Example

(i) The sequence $\langle S_n \rangle = \langle 1,3,5,\ldots \rangle$ is not a bounded sequence.
For, The range of $\langle S_n \rangle = \{1,3,5,\ldots\}$ is not bounded and so $\langle S_n \rangle$ is not bounded.

(ii) The sequence $\langle S_n \rangle = \langle -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16} \ldots \rangle$ is a bounded sequence.
For, The range of $\langle S_n \rangle = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16} \ldots\right\}$ is bounded and so $\langle S_n \rangle$ is bounded.

(iii) The sequence $\langle S_n \rangle = \langle 1,0,1,0,\ldots \rangle$ is a bounded sequence.
For, The range of $\langle S_n \rangle = \{0,1\}$ is bounded and so $\langle S_n \rangle$ is a bounded sequence.

2.7 Converges

The sequence $\langle a_n : n \in N \rangle$ of real numbers converges to $b \in R$, or, equivalently b the limit of the sequence $\langle a_n : n \in N \rangle$ if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $n > n_0$ implies $|a_n - b| < \varepsilon$.

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2.8 Cauchy Sequence

A sequence $\langle a_n : n \in N \rangle$ of real numbers is a Cauchy sequence if and only if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $n, m > n_0$ implies $|a_n - a_m| < \varepsilon$.In other words, a sequence is a Cauchy sequence if and only if the terms of sequence become arbitrarily close to each other as gets n large [4].

2.8.1 Example

Show that every convergent sequence is a Cauchy sequence.

Solution

Let $\langle a_n \rangle$is a convergent sequence with b.

$\forall \varepsilon > 0, \exists n_o \in N : n > n_0 \implies |a_n - b| < \frac{\varepsilon}{2}$.

$m > n_0 \implies |a_m - b| < \frac{\varepsilon}{2}$.

For $n, m > n_0 \implies |a_n - a_m|$

$= |a_n - b + b - a_m|$

$\leq |a_n - b| + |b - a_m|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$.

Then $\langle a_n \rangle$ is a Cauchy sequence.

2.8.2 Example

Show that every Cauchy sequence $\langle a_n \rangle$ of real numbers is bounded.

Solution

Let $\langle a_n \rangle$be a Cauchy sequence.

$Let\ \varepsilon = 1, \exists n_o \in N : n, m > n_0 \implies |a_n - a_m| < 1$.

$m \geq n_0 \implies |a_n - a_m| < 1$.

i.e, -1$< a_n - a_{n_0} < 1$

$a_{n_0} - 1 < a_n < a_{n_0} + 1$

Let $\propto = \max(a_1, a_2, \dots, a_{n_0}, a_{n_0} + 1)$

$\beta = \min(a_1, a_2, \dots, a_{n_0}, a_{n_0} - 1)$.

Then $\propto$ is an upper bound and β is a lower bound for the range $\{a_n\}$ of the sequence $\langle a_n \rangle$. Hence $\langle a_n \rangle$ is bounded.

2.9 Some Properties on Topological Spaces

Let X be a non-empty set. The class τ of subsets of X is a topology on X if and only if τ satisfies the following axioms.

$[O_1]$ $\emptyset \in \tau, X \in \tau$.

$[O_2]$ The union of any number of sets in τ belongs to τ.

$[O_3]$ The intersection of any two sets in belongs to τ.

The numbers of τ are called open sets and the pair (X, τ) is called a topological space [1-3].

2.9.1 Usual Topology

Let μ denote the class of all open sets of real numbers. Then μ is a topology on R, called usual topology on R.

2.9.2 Example

Consider the classes of subsets of $X = \{a, b, c, d, e\}$.

$\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

$\tau_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

$\tau_3 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$.

Solution

Observe that τ is a topology on X, since it satisfies the three axioms.

For, since X and $\emptyset$ belong to τ_1, the axiom $[O_1]$ is satisfied.

Since the union of any number of sets in τ_1 belongs to τ_1, the axiom $[O_2]$ is satisfied.

Since the intersection of any two sets in τ_1 belongs to τ_1, the axiom $[O_3]$ is satisfied.

For τ_2, τ_2 is not a topology on X, since $\{a, c, d\} \in \tau_2, \{b, c, d\} \in \tau_2$, but $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \tau_2, i.e, \tau_2$does not satisfy the axiom $[O_2]$.

For τ_3, τ_3 is not a topology on X, since $\{a, c, d\} \in \tau_3, \{a, b, d, e\} \in \tau_3$, but $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \tau_3, i.e., \tau_3$ does not satisfy the axiom $[O_3]$.

2.10 Discrete Topological Space and Indiscrete Topological Space

Let D be denote the class of all subsets of X. Then D satisfies the axioms for a topology on X. This topology is called the discrete topology and (X, D) is called a discrete topological space.

The class $J = \{X, \emptyset\}$ is a topology. It is called the indiscrete topology and (X, J) is called a indiscrete topological space [4].

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2.10.1 Example

The intersection $\tau_1 \cap \tau_2$ of any two topologies τ_1 *and* τ_2 on X is also a topology on X.

Solution

Since τ_1 *and* τ_2 are topologies on X, X and $\emptyset$ each belongs to both τ_1 *and* τ_2.

Hence X *and* $\emptyset$ each belongs to $\tau_1 \cap \tau_2$. The axiom $[O_1]$ is satisfied.

Let $A_i \in \tau_1 \cap \tau_2$, then $A_i \in \tau_1$ and $A_i \in \tau_2$ for every i. Since τ_1 and τ_2 are topologies, $\bigcup_i A_i \in \tau_1$ and $\bigcup_i A_i \in \tau_2$.

Then $\bigcup_i A_i \in \tau_1 \cap \tau_2$. The axiom $[O_2]$ is satisfied.

If $G, H \in \tau_1 \cap \tau_2$, then $G, H \in \tau_1$ and $G, H \in \tau_2$. Since τ_1 and τ_2 are topologies, $G \cap H \in \tau_1$ and $G \cap H \in \tau_2$.

Then $G \cap H \in \tau_1 \cap \tau_2$. The axiom $[O_3]$ is satisfied. Thus $\tau_1 \cap \tau_2$ is a topology on X.

But, the union of topologies need not be a topology.

Counter example,

$X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}\}, \tau_2 = \{X, \emptyset, \{b\}\}$.

τ_1 and τ_2 are topologies.

$\tau_1 \cup \tau_2 = \{X, \emptyset, \{a\}, \{b\}\}$.

$\{a\} \in \tau_1 \cup \tau_2$

$\{b\} \in \tau_1 \cup \tau_2$

$\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$

Therefore $[O_2]$ is not satisfied.

Therefore $\tau_1 \cup \tau_2$ is not a topology.

In topology, a discrete space is a particularly simple example of a topological space or similar structure, one in which the points form a discontinuous sequence, meaning they are isolated from each other in a certain sense. The discrete topology is the finest topology that can be given on a set, i.e., it defines all subsets as open sets. In particular, each singleton is an open set in the discrete topology [5].

3. CONCLUSIONS

We have discussed the basic properties open sets, closed sets and some characterizations of these sets on the topology of the line and plane. Moreover, it is also mentioned in this paper that a topological spaces. Some illustrative examples have been explored in order to distinguish between their properties. It has been vividly discussed in this paper. I hope thatthese notes help; please do let me know if anything requires clarification.

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