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Application Of Z-Transformation On Gambler's Ruin In Probability Theory And Other Result

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ABSTRACT

Z-transformation is one of the most popular tools to solve Mathematical, Physical, biological, electronic, control and signal processing systems and difference equation. The purpose of this research paper is to discuss the application of z-transformation on Gambler's ruin in probability theory. In this paper Gambler's ruin model in probability theory and compound interest model are discussed via z-transformation.[1, 2]

Keywords

Z –transformation, Gambler's ruin, Compound Interest.

1. INTRODUCTION

Gambler's ruin Problem was first solved by Pascal and Fermat and published by Huygens since 1657. It was studied and extended by many probabilists in early years and thus, it became an important problem in probability history, introducing many new concepts. Gambler's ruin problem was proposed by Pascal to Fermat in 1656.It is two years later than the correspondence between Pascal and Fermat on the "Problem of points". It is not hard to imagine that it was an interesting and difficult problem among gambler's at that time, because we can see similar situations in the modern finance. These days people are interested in various probabilities related to many financial instruments. In particular, when people make up new financial instruments, such as a new type of stock option, they calculate many types of probabilities that can show how profitable it is presumably, Pascal did not have any practical concern about this problem, but he considered it is a very difficult and interesting problem in the mathematical perspective. The two mathematicians, Pascal and Fermat, solved this problem independently and presumably with different method.[3, 4]

2. APPLICATION OF Z-TRANSFORMATION ON GAMBLER'S RUIN IN PROBABILITY THEORY

2.1 Definition of Gambler's ruin

A gambler plays a sequence of games against an adversary .If β is the probability that the gambler wins dollar 1 in any given game then (1- β) will be the probability of him losing dollar 1.

He quits the game if he either wins a prescribed amount of N dollar or loses all his money .If he loses all his money then we say that he has been ruined[5].

2.2 A solution of Gambler's ruin By Z-transformation

Let $P(n)$ denotes the probability that the gambler will be ruined if he starts gambling with n dollar.

The second order difference equation[6, 7] is given by

$$P(n+2)-\frac{1}{\beta}P(n+1)+\frac{1-\beta}{\beta}P(n)=0 \quad \text{With } n=0,1,2,\cdots\cdots\cdots N$$

Or, $P_{n+2}-\frac{1}{\beta}P_{n+1}+\frac{1-\beta}{\beta}P_n=0$

Taking z-transformation both side

$$Z(P_{n+2})-\frac{1}{\beta}Z(P_{n+1})+\frac{1-\beta}{\beta}Z(P_n)=Z(0)$$

$$\Rightarrow [z^2\,P(z)-z^2P(0)-zP(1)]-\frac{1}{\beta}[zP(z)-zP(0)]+\frac{1-\beta}{\beta}P(z)=0$$

$$\Rightarrow P(z)[z^2-\frac{1}{\beta}z+\frac{1-\beta}{\beta}]=z^2P(0)+zP(1)-z\frac{P(0)}{\beta}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1.1)$$

We know that probability of ruin starting with 0 dollar is 1 and hence $\mathrm{P}(0)=1$ and If the player has N dollars then He quits and cannot be ruined so that $\mathrm{P(N)}=0$.

Let $\mathrm{P}(1)=a$

Equation (1.1) $\Rightarrow P(z)[z^2-\frac{1}{\beta}z+\frac{1-\beta}{\beta}]=z^2.1+z.a-z\frac{1}{\beta}\cdots\cdots\cdots\cdots\cdots\cdots(1.2)$

Now two cases arise

Case (i) If $\beta=\frac{1}{2}$, Equation (1.2)$\Rightarrow P(z)[z^2-2z+1]=z^2+z.a-2z$

$$\Rightarrow P(z)=\frac{z^2}{(z-1)^2}+(a-2)\frac{z}{(z-1)^2}$$

$$\Rightarrow P(z)=Z[\frac{1}{z-1}+\frac{1}{(z-1)^2}]+(a-2)\frac{z}{(z-1)^2}=\Rightarrow P(z)=\frac{z}{z-1}+(a-1)\frac{z}{(z-1)^2}$$

Now taking inverse Z-transformation on both side

$$P(\mathrm{n})=Z^{-1}(\frac{z}{z-1})+(a-1)\,\mathrm{Z}^{-1}(\frac{z}{(z-1)^2})$$

$$\Rightarrow \mathrm{P(n)}=1+(\mathrm{a}-1)\,\mathrm{n}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1.3)$$

If $\mathrm{n}=\mathrm{N}$ Then $\mathrm{P(N)}=1+(\mathrm{a}-1)\,\mathrm{N}\Rightarrow 0=1+(\mathrm{a}-1)\,\mathrm{N}\Rightarrow(\mathrm{a}-1)=\frac{-1}{N}$

Now Equation (1.3) $\Rightarrow P(\mathrm{n}) = 1 - \frac{n}{N}$ ....(1.4)

Case (ii) If $\beta \neq \frac{1}{2}$, Let $\beta = \frac{1}{3}$, Equation (1.2) $\Rightarrow P(z)(z^2 - 3z + 2) = z^2 + za - 3z$

$$\Rightarrow P(z) = \frac{z^2}{(z-1)(z-2)} + (a-3)\frac{z}{(z-1)(z-2)}$$

$$\Rightarrow P(z) = z[\frac{2}{z-2} - \frac{1}{z-1}] + (a-3)z[\frac{1}{z-2} - \frac{1}{z-1}]$$

$$\Rightarrow \mathrm{P(z)} = (\mathrm{a}-1)\frac{z}{z-2} - (a-2)\frac{z}{z-1}$$

Now, taking inverse Z- transformation, we get

$$\mathrm{P(n)} = (\mathrm{a}-1)\,\mathrm{Z}^{-1}(\frac{z}{z-2}) - (a-2)\,\mathrm{Z}^{-1}(\frac{z}{z-1})$$

$$\Rightarrow \mathrm{P(n)} = (\mathrm{a}-1)2^n - (a-2) \cdots\cdots\cdots\cdots\cdots (1.5)$$

Put $n = N$ equation $(1.5) \Rightarrow \mathrm{P(N)} = (\mathrm{a}-1)2^N - (a-2)$

$$\Rightarrow 0 = \mathrm{a}\,2^N - 2^N - a + 2$$

$$\Rightarrow a = \frac{2^N - 2}{2^N - 1}$$

Equation $(1.5) \Rightarrow \mathrm{P(n)} = (\frac{2^N - 2}{2^N - 1} - 1)2^N - (\frac{2^N - 2}{2^N - 1} - 2)$

$$\Rightarrow \mathrm{P(n)} = (\frac{2^N - 2^n}{2^N - 1})$$

Example (1):-If $\beta = \frac{1}{2}$ and gambler starts with 30 dollar with $N = 1000$ then

That is, his ruin is almost certain.

In general, If the gambler plays a long series of games which can be modelled here as $N \to \infty$ then he will be ruined almost certainly even if the game is fare ($\beta = \frac{1}{2}$).

4. APPLICATION OF Z-TRANSFORMATION ON COMPOUND INTEREST MODEL

4.1Compoud interest model

Assume that A_n denote the amount in the account at the end of the nth year that bears b percent interest per year then we have $A_{n+1} = A_n + 0.0bA_n \Rightarrow A_{n+1} - (1 + 0.0b)A_n = 0$ which is the compound interest model.

4.2 A solution of compound interest model by z-transformation

Compound interest model is

$$A_{n+1} - (1+0.0b)A_n = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1.6)$$

Applying Z-transformation throughout, we get

$$Z(A_{n+1}) - (1+0.0b)\,\mathrm{Z}(A_n) = Z(0)$$

$$\Rightarrow zA(z) - zA(0) - (1+0.0b)A(z) = 0$$

$$\Rightarrow (z-1.0b)A(z) = ZA(0)$$

Let $\mathrm{A}(0) = \mathrm{P}$ then $\mathrm{A(z)} = \dfrac{zP}{z-1.0\mathrm{b}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1.7)$

Taking inverse z-transformation both side, we get

$$Z^{-1}(A(z)) = \mathrm{PZ}^{-1}\left(\frac{z}{z-1.0b}\right)$$

$$\Rightarrow \mathrm{A(n)} = \mathrm{P}(1.0\,\mathrm{b})^n$$

5. CONCLUSION

We achieved same result of compound interest by z-transformation.

Example (1) If 1000 dollars is put into an account bearing 6 percent interest then at the end of 10 years will be $A_{10} = 1000(1.06)^{10} = 1000(1.79084769654) = 1790.84769654\,\text{dollars}$.

(2) If 1000 dollars is put into an account with interest 0.05 compounded monthly then the monthly interest is $\dfrac{0.05}{12} = 0.00416666 = 0.00417$ and at the end of 10 years there will have been 120 interest periods. therefore $A_{120} = 1000(1.00417)^{120} = 1000(1.64766569746) = 1647.66569746$ dollars.

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