Variational Iteration Method for First and Second Order Ordinary Differential Equations using First Kind Chebychev Polynomials

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ABSTRACT

In this research work, we focused on the application of the first kind Chebychev polynomials as basis functions for the numerical solution of first and second order ordinary differential equations. For this purpose, the variational iteration method (VIM) was adopted as an iterative scheme to generate the required approximate solutions. The VIM with the Chebychev polynomials was applied to some selected linear and nonlinear problems for experimentations, and the resulting numerical evidence shows that it is effective and accurate.

Keywords: Differential equation, Chebychev polynomials, Iteration method

1. INTRODUCTION

Let the generalized form of a differential equation be given as

$$L[ux] = gx, \ ua_1 = a, \ ua_2 = b, \qquad (1)$$

where L is considered as differential operator, $ua_1 = a, \ ua_2 = b,$ are boundary or initial conditions.

Equations of the type (1) are often applied in the construction and development of most mathematical models such as predictive control in AP monitor Hedengen et. al. [1], temperature distribution in cylindrical conductor Fortini et. al. [2], dynamic optimization Cizinar et. al. [3], etc.

Modeling is the bridge between the subject and real-life situations for student realization. Differential equations model real-life situations, and provide the real-life answers with the help of computer calculations and graphics. Thus, solving these problems (1) is deemed necessary to provide the real-life answers. Consequently, many methods have been proposed the years for solving problems such as (1), which can be seen in [4-8].

Here in this research, we employ the numerical scheme called the variational iteration method for seeking the approximate solutions to equation (1) using Chebychev polynomials of the first kind as trial functions.

For this purpose, we define an approximation to (1) as

$$u_n x = \sum_{r=0}^{n} \alpha_r \, T_r x, -1 \le x \le 1, \qquad (2)$$

where $\alpha_r, r = 0(1)n$ are constants to be determined, $T_r x, r = 0(1)n$ are the Chebychev polynomials defined in the interval $-1 \leq x \leq 1$.

2. CHEBYCHEV POLYNOMIALS

The Chebychev polynomial of the first kind is defined as [9]

$$T_r x = cosrcos^{-1}x = \sum_{i=0}^{n} C_r^{(n)} x^i, \quad -1 \leq x \leq 1, \tag{3}$$

with

$$T_{r+1}x = 2xT_r x - T_{r-1}x, \tag{4}$$

satisfying the conditions

$$T_0 x = 1 \text{ and } T_1 x = x.$$

Now, if $x \in [-1,1]$ is mapped objectively to $a \leq x \leq b$, then equation (3) becomes

$$T_{r+1}^* x = 2xT_r^*(x) - T_{r-1}^* x, \tag{5}$$

satisfying the conditions

$$T_0^*(x) = 1 \text{ and } T_1^*(x) = \frac{2x-a-b}{b-a}.$$

Equation (5) is called the nth degree shifted Chebychev polynomials. Thus, the first seven (7) Chebychev polynomials are presented below:

$$T_0 x = 1$$
$$T_1 x = x$$
$$T_2 x = 2x^2 - 1$$
$$T_3 x = 4x^3 - 3x$$
$$T_4 x = 8x^4 - 8x^2 + 1$$
$$T_5 x = 16x^5 - 20x^3 + 5x$$
$$T_6 x = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7 x = 64x^7 - 112x^5 + 56x^3 - 7x$$

3. BASIC IDEAS OF VARIATIONAL ITERATION METHOD

The variational iteration method (VIM) established by Ji-Huan [10-16] is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact solution exists.

Now, the VIM involves the construction of a correction functional for Equation (1) is as follow:

$$u_{i+1}x = u_i x + \int_0^x \lambda(s)Lu_i(s) - gs \, ds, i \geq 0, \tag{6}$$

where $\lambda(s)$ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. Also, the Lagrange multiplier $\lambda(s)$ can be estimated using the formula [12]

$$\lambda s = -1^n \frac{(s-x)^{(n-1)}}{n-1!}, \tag{7}$$

where n in Equation (7) is the order of the derivative.

3.1: Determination of Initial Approximation Using Chebychev polynomials of the First Kind

Let n = 0 in equation (3), then an initial approximation of the iterative scheme is given as

$$u_{n-1}x = \sum_{r=0}^{n-1} \alpha_r T_r x, \quad x \in [-1,1], \tag{8}$$

where $\alpha_r, r = 0 \dots (n-1)$ are constants to be determined, $T_r x, r = 0 \dots (n-1)$ are the Chebychev polynomials defined in the interval $-1 \leq x \leq 1$, and n is the order of equation (1).

Now, when $n = 1$ in (8), we have,

$$u_0 x = \alpha_0 \varphi_0 x. \tag{9}$$

At the condition $ua_1 = a$, we have the above equation (9) as

$$u_0 a_1 = \alpha_0 = a.$$

Thus, the variational iterative scheme for (1) when $n = 1$ is given as

$$\begin{cases} u_0 x = a, \\ u_{i+1}x = u_i x - \int_0^x Lu_i(s) \ - gs \, ds, \ i \geq 0 \end{cases} \tag{10}$$

Similarly, Now, when $n = 2$ in (8), we have,

$$u_1 x = \alpha_0 \varphi_0 x + \alpha_1 \varphi_1 x \tag{11}$$

At the condition $ua_1 = a$, we have equation (11) as

$$\alpha_0 + \alpha_1 a_1 = a \tag{12}$$

Also, at the condition $ua_2 = b$, we have equation (3.26) as

$$\alpha_0 + \alpha_1 a_2 = b \tag{13}$$

On solving (12) and (13), we obtain,

$$\alpha_1 = \frac{a-b}{a_1-a_2}, \quad \alpha_0 = \frac{a_1 b - a a_2}{a_1 - a_2}.$$

Thus, the variational iterative scheme for (1) when $n = 2$ is given as

$$\begin{cases} u_1 x = \frac{a-b}{a_1-a_2} + \frac{a_1 b - a a_2}{a_1 - a_2} x \\ u_{i+1}x = u_i x + \int_0^x (s-x)Lu_i(s) \ - gs \, ds, \ i \geq 1 \end{cases} \tag{14}$$

Suppose that the conditions in (1) are mixed, that is, $a_1 = a \ u'a_2 = b$, then the variational iteration scheme for (1) when $n = 2$ becomes

$$\begin{cases} u_1 x = a - a_1 b + bx, \\ u_{i+1}x = u_i x + \int_0^x (s-x)Lu_i(s) \ - gs \, ds, \ i \geq 1 \end{cases} \tag{15}$$

4.1   Numerical Experiments

In this section, we present some numerical experimentation for obtaining the approximate solution of first and second order ordinary differential equations in a variational iteration method employing the Chebychev polynomials of the first polynomials as basis functions.

The error formulation for these problems is defined explicitly as

$$e_r = |ux - u_i(x)|, \qquad r = 1,2,3 \dots, \tag{16}$$

where ux is the analytic or exact solution as available in literature, and $u_i(x)$ is the computed solution using the results in (10), (14) and (15) respectively. Results obtained are presented in tables for convergence observation.

Example 4.1 [15]

Solve the following initial value problem up to third approximation

$$\frac{du}{dx} = 2u - 2x^2 - 3, \qquad u = 2 \text{ when } x = 0. \tag{17}$$

The analytic solution is given as $ux = x^2 + x + 2$.

Since (17) is first order ODE, using the iterative scheme in (10) with the aid of Maple 18 software, we obtain the following approximations:

$$u_0 x = 2$$

$$u_1 x = 2 - \frac{2}{3}x^3 + x$$

$$u_2 x = 2 - \frac{2}{3}x^3 + x - \frac{1}{3}x^4 + x^2$$

$$u_3 x = 2 - \frac{2}{15}x^3 + x - \frac{1}{3}x^4 + x^2$$

$$\vdots$$

Computational results for Example 4.1 is given in table 1 below

Table 1: Comparison of computed results and analytic solution for Example 4.1

x	$u(x)$	$u_1(x$	$e_1(x)$	$u_2(x)$	$e_2(x)$	$u_3(x)$	$e_3(x)$
0.00	2.0000000	2.0000000	0.000e+00	2.0000000	0.000e+00	2.000000	0.0000e+00
0.10	2.1100000	2.0993333	1.0667e-02	2.1093000	7.0000e-04	2.1099653	3.4667e-05
0.20	2.2400000	2.1946667	4.5333e-02	2.2341333	5.8667e-03	2.2394240	5.7600e-04
0.30	2.3900000	2.2820000	1.0800e-01	2.3693000	2.0700e-02	2.3869760	3.0240e-03
0.40	2.5600000	2.3573333	2.0267e-01	2.5088000	5.1200e-02	2.5501013	9.8987e-03
0.50	2.7500000	2.4166667	3.3333e-01	2.6458333	1.0417e-01	2.7250000	2.5000e-02

Example 4.2 [15]

Consider the linear initial value problem in second order ordinary differential equation

$$\frac{d^2u}{dx^2} = 2ux + 1 \text{ given that } u0 = 1, u'0 = 0. \tag{18}$$

The analytic solution is given as $ux = \dfrac{3}{4} e^{\sqrt{2}\, x} + \dfrac{3}{4} e^{-\sqrt{2}\, x} - \dfrac{1}{2}$.

Since (18) is second order ODE, using the iterative scheme in (14) with the aid of Maple 18 software, we obtain the following approximations

$$u_0 x = 1$$

$$u_1 x = 1 + \frac{3}{2}x^2$$

$$u_2 x = 1 + \frac{3}{2}x^2 + \frac{1}{4}x^4 .$$

$$u_3 x = 1 + \frac{3}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{60}x^6 .$$

$$u_4x = \frac{1 + \frac{3}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{60}x^6 + \frac{1}{60}x^{10}}{}$$

⋮

Computational results for Example 4.2 is given in table 2 below.

Table 2: Comparison of maximum errors obtained from Example 4.2

x	$e_1(x)$	$e_2(x)$	$e_3(x)$	$e_4(x)$
0.00	0.0000+e00	0.0000+e00	0.000e+00	0.000e+00
0.10	2.5016e-05	1.6000e-08	1.0000e-09	1.0000e-09
0.20	4.0107e-04	1.0680e-06	1.0000e-09	0.000e+00
0.30	2.0372e-03	1.2189e-05	3.9000e-08	5.9000e-08
0.40	6.4687e-03	6.8658e-05	3.9100e-07	1.3560e-06
0.50	1.5888e-02	2.6276e-04	2.3380e-06	1.3938e-05

5. DISCUSSION OF RESULTS

We have successfully applied the variational iteration method (VIM) for the numerical solution of first and second order differential equations (ODE) with first kind Chebychev polynomials as basis functions. The results obtained above show that the Chebychev polynomials as basis/trial functions provide an excellent convergence of the solution of first and second order ordinary differential equations as shown in the tables 1 and 2 above. We obtained satisfactory resuts because of the excellent convergence rate of the VIM with MNPs as basis functions, which give same approximation as that of Mamadu-Njoseh polynomials.

6. CONCLUSION

This reseach has considered the numerical treatment of first kind Chebychev polynomials as basis functions for the solution of first and second order differential equations. So far, it has been well established that first kind Chebychev polynomials can be employed as basis functions in solving differential equations. It is also evident that the numerical scheme adopted for the reseach (VIM) offers several advantages which iclude:

i. initial/boundary conditions can be chosen freely with some unknown parameters;

ii. unknown parameters initial/boundary condition can be easily identified; and

iii. the calculation is simple and straightforward.

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